LECTURE 1

Symplectic reduction, II

1.1. Resolution of singularities

Recall that quotients are the most well-behaved when the group action is free. We say a point $x \in X$ is ***regular*** if the orbit $\mathbb{O}_x$ is closed and the stabilizer G_x is trivial. The set of regular elements $X^{reg} \subset X$ is open and G-invariant. Moreover, we can consider the set of regular orbits

(1) $$(X /\!/ G)^{reg} \subset X /\!/ G.$$

From the Luna slice theorem it follows that the set of regular points and the set of regular orbits $(X /\!/ G)^{reg}$ is open in $X /\!/ G$.

THEOREM 1.1.1. *The following hold.*

1. *If $x \in X^{reg}$ then x is χ-stable for any character χ.*
2. *The subspace of regular orbits $(X /\!/ G)^{reg}$ is nonsingular. Let $\pi\colon X /\!/_\chi G \to X /\!/ G$ be the canonical map, then*

(2) $$(X /\!/_\chi G)^{reg} \overset{\text{def}}{=} \pi^{-1}(X /\!/ G)^{reg}$$

is also nonsingular and the map

(3) $$\pi\colon (X /\!/_\chi G)^{reg} \to (X /\!/ G)^{reg}$$

is an isomorphism.

Recall that a morphism $\pi\colon X \to Y$ is called a resolution of singularities if X is non-singular and π is proper and birational. Being birational means that there exists an open dense subset $Y_0 \subset Y$ such that $\pi^{-1}(Y_0)$ is dense in X and the restriction $\pi\colon \pi^{-1}(Y_0) \to Y_0$ is an isomorphism.

As a consequence of the twisted GIT theorem above we have the following result on resolutions of singularities.

Corollary 1.1.2. *Suppose that X^{reg} is nonempty and that $X /\!/_\chi G$ is nonsingular and connected. Then*

(4) $$\pi\colon X /\!/_\chi G \to X /\!/ G$$

is a resolution of singularities.

PROOF. By the theorem we see that $(X /\!/_\chi G)^{reg}$ is nonempty and open in $X /\!/_\chi G$. □

1.2. Type A_1 singularity

Consider the affine subvariety

$$X \stackrel{\text{def}}{=} \{(i,j) \mid ij = 0\} \subset (\mathbf{C}^2)^* \times \mathbf{C}^2 \simeq \mathbf{C}^4. \tag{5}$$

There is an action of $\mathbf{C}^\times$ on X defined by

$$\lambda \cdot (i,j) = (\lambda i, \lambda^{-1} j). \tag{6}$$

We first consider the affine GIT quotient $X /\!/ \mathbf{C}^\times$.

Let $\mathrm{End}_0(\mathbf{C}^2) \subset \mathrm{End}(\mathbf{C}^2)$ denote the three-dimensional vector space of traceless 2×2 matrices. Every such matrix has the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{7}$$

where $a, b, c \in \mathbf{C}$.

Lemma 1.2.1. *Let*

$$Q = \{A \mid \det(A) = 0\} \subset \mathrm{End}_0(\mathbf{C}^2) \tag{8}$$

be the singular quadric defined by the equation

$$a^2 + bc = 0 \tag{9}$$

in $\mathbf{A}^3 = \mathrm{Spec}\, \mathbf{C}[a,b,c]$. *The map*

$$\Phi \colon X /\!/ \mathbf{C}^\times \xrightarrow{\simeq} Q \tag{10}$$

defined by $\Phi(i,j) = ji$ *is an isomorphism of affine varieties.*

Proof. It is easy to see that $\widetilde{\Phi} \colon X \to Q$ is well-defined since the condition $ij = 0$ implies that $\mathrm{tr}(ji) = \det(ji) = 0$. Clearly $\widetilde{\Phi}$ descends to the map $\Phi \colon X /\!/ \mathbf{C}^\times \to Q$. The inverse sends a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in Q \tag{11}$$

to the pair $[(i_A, j_A)]$ where $i_A = \begin{pmatrix} x & y \end{pmatrix}$ and $j_A = \begin{pmatrix} z & w \end{pmatrix}^t$. If $a = 0$ then either b or c must be zero; in the case $b = 0$ then we set $x = 0, y = 1, z = c, w = 0$. It is easy to see that this is well-defined up to the $\mathbf{C}^\times$ action. If $a \neq 0$ then set $x = a, y = b, z = 1, w = c/a$. □

Points in X with zero stabilizer are those where both i, j are nonzero. This is equivalent to the condition that $ji \neq 0$. Thus

$$(X /\!/ \mathbf{C}^\times)^{reg} \simeq Q \setminus \{0\}. \tag{12}$$

This is certainly a nonsingular variety.

Let's now consider the twisted GIT quotient. For the character let's take the identity morphism $\chi = \mathbb{1} \colon \mathbf{C}^\times \to \mathbf{C}^\times$. Suppose that $(i,j;\mu) \in X \times \mathbf{C}$ thought of as elements of the trivial line bundle over X. Then using the action in (**??**) we have, for $\lambda \in \mathbf{C}^\times$

$$\lambda \cdot (i,j;\mu) = (\lambda i, \lambda^{-1} j; \lambda^{-1} \mu). \tag{13}$$

Notice that if $j = 0$ and $i \neq 0, \mu \neq 0$ then

$$\mathcal{O}_{(i,0;\mu)} = \{(a, 0; \alpha) \mid a \neq 0, \alpha \neq 0\} \subset X \times \mathbf{C}. \tag{14}$$

Thus $(i, 0)$ is not a semi-stable point. Conversely we see that as long as $j \neq 0$ then the point (i, j) is semi-stable.

Proposition 1.2.2. *There is an isomorphism*

$$X /\!/_\chi \mathbf{C}^\times \simeq \mathrm{T}^*\mathbf{P}^1. \tag{15}$$

PROOF. From the characterization of semi-stable elements we see that

$$X /\!/_\chi \mathbf{C}^\times = \{(L, i) \mid L \subset \mathbf{C}^2 \text{ line}, \quad i|_L = 0\}. \tag{16}$$

Thus, there is a canonical map $X /\!/_\chi \mathbf{C}^\times \to \mathbf{P}^1$ defined by $(L, i) \mapsto L$. This map endows $X /\!/_\chi \mathbf{C}^\times$ with the structure of a line bundle over $\mathbf{P}^1$. We will identify this line bundle.

Let $L \subset \mathbf{C}^2$ be a line. A choice of a nonzero vector $v \in L$ determines an isomorphism $T_L\mathbf{P}^1 \simeq_v \mathbf{C}^2/L$. Hence

$$\mathrm{T}^*_L\mathbf{P}^1 \simeq \{i \colon \mathbf{C}^2 \to L \mid i|_L = 0\}. \tag{17}$$

This isomorphism is independent of the choice of nonzero $v \in L$. Thus $X /\!/_\chi \mathbf{C}^\times \simeq \mathrm{T}^*\mathbf{P}^1$. □

From this discussion we conclude that there is a resolution of singularities

$$\pi \colon \mathrm{T}^*\mathbf{P}^1 \to Q. \tag{18}$$

This resolution is a special case of the so-called Springer resolution which we will discuss next time.

1.3. SYMPLECTIC ACTIONS

Let (M, ω) be a symplectic manifold and suppose G is acting on M.

- If M is a smooth symplectic manifold then we assume that G is a real Lie group and the action is smooth.
- If M is a symplectic algebraic variety then we assume that G is a linear algebrac group acting algebraically.

The G-action is ***symplectic*** if it preserves the symplectic form; that is for every $g \in G$ the corresponding diffeomorphism ϕ_g satisfies $\phi_g^*\omega = \omega$. Infinitesimally, this means that for every $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$ the corresponding vector field $X_\xi \in \mathrm{Vect}(M)$ satisfies

$$L_{X_\xi}\omega = 0. \tag{19}$$

Such vector fields are called symplectic vector fields; the space of all symplectic vector fields $\mathrm{Vect}_\omega(M) \subset \mathrm{Vect}(M)$ is a sub Lie algebra of the Lie algebra of all smooth vector fields. So, a symplectic action ρ of G on M determines a map of Lie algebras

$$D\rho \colon \mathfrak{g} \to \mathrm{Vect}_\omega(M). \tag{20}$$

Locally, in the C^∞ world, every symplectic vector field is determined by a function. Indeed the symplectic form determines an isomorphism of $\mathrm{Vect}_\omega(M)$ with the space of *closed* one-forms $\Omega^{1,cl}(M)$. But, by the C^∞-Poincaré lemma, locally every closed one-form is exact. So, given a symplectic vector field ξ we can locally find a function $H \in C^\infty(M)$ such that

$$\xi = X_H \tag{21}$$

where $X_H = \omega^{-1}(\mathrm{d}H)$ is the Hamiltonian vector field corresponding to H.

The symplectic form ω determines a Poisson bracket $\{-,-\}$ on the commutative algebra of functions. The map

$$\{-,-\}\colon C^\infty(M) \to \mathrm{Vect}_\omega(M) \tag{22}$$

is a map of Lie algebras. Every constant function is sent to the zero vector field. Using this, one can show that there is a central extension of Lie algebras

$$0 \to \mathbf{C} \to C^\infty(M) \to \mathrm{Vect}_\omega(M) \to 0. \tag{23}$$

This extension may not be split.

Definition 1.3.1. A symplectic action ρ of G on M is ***Hamiltonian*** if there exists a G-equivariant map

$$\mu\colon M \to \mathfrak{g}^* \tag{24}$$

such that

(1) For any $a \in \mathfrak{g}$ the function

$$H_a(x) = \langle \mu(x), a \rangle \tag{25}$$

is a Hamiltonian function for the vector field $\xi_a = D\rho(a)$.

(2) The assignment $a \mapsto H_a$ is a map of Lie algebras $\mathfrak{g} \to C^\infty(M)$.

Example 1.3.2. Suppose that V is a vector space equipped with a nondegenerate skew-symmetric bilinear form $\omega \in \wedge^2 V^*$. Thus (V, ω) is a symplectic vector space. Suppose that $G \subset Sp(V)$ acts on V in a way that preserves ω. Such an action is always Hamiltonian. Indeed, define

$$\mu\colon V \to \mathfrak{g}^* \tag{26}$$

by the rule

$$\langle \mu(v), a \rangle = \frac{1}{2}\omega(v, a \cdot v), \quad \text{for all} \quad a \in \mathfrak{g}. \tag{27}$$

Here $\langle -,- \rangle$ denotes the canonical pairing between $\mathfrak{g}$ and its dual.

Example 1.3.3. Suppose that N is a smooth manifold with a G-action. Then G extends to a Hamiltonian action on T^*N with moment map defined by

$$\langle \mu(x, \eta), a \rangle_{\mathfrak{g}} = \langle \eta, \xi_a(x) \rangle \tag{28}$$

where the right-hand side is the canonical pairing between one-forms and vector fields.

A nice way to summarize the structure of the moment map is the following. We have pointed out that the algebra of functions on a symplectic manifold is equipped with a Poisson bracket. More generally, we can consider manifolds (which are not necessarily symplectic) whose functions are equipped with a Poisson bracket—such a manifold is a Poisson manifold. For any Lie algebra $\mathfrak{g}$ its dual $\mathfrak{g}^*$, thought of as a vector space, satisfies

$$\mathcal{O}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}). \tag{29}$$

The Lie bracket $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ determines a Poisson bracket on $\mathrm{Sym}(\mathfrak{g})$. Thus, $\mathfrak{g}^*$ has the canonical structure of a Poisson manifold.

THEOREM 1.3.4. *Let M be a symplectic manifold with a Hamiltonian G action. Then the moment map $\mu\colon M \to \mathfrak{g}^*$ is a map of Poisson manifolds.*