LECTURE 3

Symplectic geometry, III

In this lecture we introduce the important construction of Hamiltonian reduction of a Hamiltonian action on a symplectic manifold.

3.1. Hamiltonian actions

Last time we built up towards the following definition.

Definition 3.1.1. A symplectic action ρ of G on M is ***Hamiltonian*** if there exists a G-equivariant map

$$\mu\colon M \to \mathfrak{g}^* \tag{1}$$

such that

(1) For any $a \in \mathfrak{g}$ the function

$$H_a(x) = \langle \mu(x), a \rangle \tag{2}$$

is a Hamiltonian function for the vector field $\xi_a = D\rho(a)$.

(2) The assignment $a \mapsto H_a$ is a map of Lie algebras $\mathfrak{g} \to C^\infty(M)$.

Example 3.1.2. Suppose that V is a vector space equipped with a nondegenerate skew-symmetric bilinear form $\omega \in \wedge^2 V^*$. Thus (V, ω) is a symplectic vector space. Suppose that $G \subset Sp(V)$ acts on V in a way that preserves ω. Such an action is always Hamiltonian. Indeed, define

$$\mu\colon V \to \mathfrak{g}^* \tag{3}$$

by the rule

$$\langle \mu(v), a \rangle = \frac{1}{2}\omega(v, a \cdot v), \quad \text{for all} \quad a \in \mathfrak{g}. \tag{4}$$

Here $\langle -, - \rangle$ denotes the canonical pairing between $\mathfrak{g}$ and its dual.

Example 3.1.3. Suppose that N is a smooth manifold with a G-action. Then G extends to a Hamiltonian action on T^*N with moment map defined by

$$\langle \mu(x, \eta), a \rangle_{\mathfrak{g}} = \langle \eta, \xi_a(x) \rangle \tag{5}$$

where the right-hand side is the canonical pairing between one-forms and vector fields.

Example 3.1.4. Let G be a linear algebraic group and $P \subset G$ a closed subgroup. Consider the variety $X = G/P$. There is a canonical isomorphism $\mathrm{T}_x X = g/\mathrm{Ad}_x \cdot \mathfrak{p}$ where $\mathfrak{p} = \mathrm{Lie} P$. Thus

$$\mathrm{T}^* X = \{(x, \lambda) \mid \lambda \in \mathrm{Ad}_x^* \cdot \mathfrak{p}^\perp\} \subset X \times \mathfrak{g}^* \tag{6}$$

where $\mathfrak{p}^\perp = \{\lambda \in \mathfrak{g}^* \mid \lambda|_\mathfrak{p} = 0\}$ and $\mathrm{Ad}^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ denotes the coadjoint action.

Consider the left action of G on $X = G/P$. This extends to a Hamiltonian action of G on T^*X with moment map

$$\mu(x, \lambda) = \lambda. \tag{7}$$

A nice way to summarize the structure of the moment map is the following. We have pointed out that the algebra of functions on a symplectic manifold is equipped with a Poisson bracket. More generally, we can consider manifolds (which are not necessarily symplectic) whose functions are equipped with a Poisson bracket—such a manifold is a Poisson manifold. For any Lie algebra $\mathfrak{g}$ its dual $\mathfrak{g}^*$, thought of as a vector space, satisfies

$$\mathcal{O}(\mathfrak{g}^*) = \mathrm{Sym}(\mathfrak{g}). \tag{8}$$

The Lie bracket $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ determines a Poisson bracket on $\mathrm{Sym}(\mathfrak{g})$. Thus, $\mathfrak{g}^*$ has the canonical structure of a Poisson manifold.

THEOREM 3.1.5. *Let M be a symplectic manifold with a Hamiltonian G action. Then the moment map $\mu\colon M \to \mathfrak{g}^*$ is a map of Poisson manifolds.*

3.2. HAMILTONIAN REDUCTION

Given a Hamiltonian G action on a symplectic manifold M, a natural question to ask is in what sense the quotient M/G is symplectic. Even if M/G is a smooth manifold it may not be the case that it is symplectic. For symplectic G-action there is a more refined procedure to produce a quotient which is symplectic (assuming it is a manifold). Let G be a real Lie group acting on a smooth manifold M.

THEOREM 3.2.1. *Suppose that M is symplectic with a proper Hamiltonian G-action with moment map $\mu\colon M \to \mathfrak{g}^*$. Let $p \in \mathfrak{g}^*$ such that*

- *p is a regular value of μ, so $\mu^{-1}(p)$ is a smooth submanifold of M.*
- *The stabilizer $G_p \subset G$ of p acts freely on $\mu^{-1}(p)$ so that $\mu^{-1}(p)/G_p = \mu^{-1}(\mathbb{O}_p)/G$ is a smooth manifold.*

Then $\mu^{-1}(p)/G_p$ has the canonical structure of a symplectic manifold compatible with the symplectic structure on M.

As a corollary we see that if G acts freely on M then $\mu^{-1}(0)/G$ has the canonical structure of a smooth symplectic manifold.

Example 3.2.2. Suppose that G acts on a smooth manifold N in a free and proper way and let $\mu\colon T^*N \to \mathfrak{g}$ be the moment map from example 3.1.3. Then there is a symplectomorphism

$$\mathrm{T}^*(N/G) \simeq \mu^{-1}(0)/G. \tag{9}$$

We now make the connection to the GIT quotient, which we recall makes sense in the affine algebro-geometric setting. Suppose that G is a reductive algebraic group acting on a nonsingular affine algebraic variety X. The cotangent bundle

T^*X is an affine algebraic variety equipped with a Hamiltonian action by G. Thus, there is an algebraic moment map $\mu\colon T^*X \to \mathfrak{g}^*$ and $\mu^{-1}(0)$ is an affine algebraic variety which is equipped with a G-action.

THEOREM 3.2.3. *Suppose the G-action on the affine algebraic variety X is free so that $X/G = X /\!/ G$ is a non-singular affine algebraic variety. Then for any G-invariant $p \in \mathfrak{g}^*$ the space $\mu^{-1}(p)/G$ is symplectic and there is a canonical symplectomorphism*

$$T^*(X/G) \simeq \mu^{-1}(0)/G. \tag{10}$$

In the above theorem we assume, importantly, that the action is free so that the GIT quotient agrees with the set-theoretic quotient. If the action is not free then generally speaking $X /\!/ G$ and $\mu^{-1}(0)/G$ are singular algebraic algebraic varieties. Nevertheless, in general $\mu^{-1}(0) \subset X$ is an affine algebraic variety and we can contemplate the GIT quotient

$$\mathcal{M}_0 \stackrel{\text{def}}{=} \mu^{-1}(0) /\!/ G. \tag{11}$$

More generally, for any character $\chi\colon G \to \mathbf{C}^\times$ we have the twisted GIT quotient

$$\mathcal{M}_\chi \stackrel{\text{def}}{=} \mu^{-1}(0) /\!/_\chi G. \tag{12}$$

By definition there is a proper map $\pi\colon \mathcal{M}_\chi \to \mathcal{M}_0$.

THEOREM 3.2.4. *Let X be a smooth affine algebraic variety and let G be a reductive algebra group acting on X. Then*

1. *For any character χ the variety $\mathcal{M}_\chi$ is Poisson. The morphism $\pi\colon \mathcal{M}_\chi \to \mathcal{M}_0$ is Poisson.*
2. *Let X^s be the χ-stable points so that $X^s /\!/ X /\!/_\chi G$ is a smooth subvariety. Also*

$$\mathcal{M}_\chi^s \stackrel{\text{def}}{=} (\mu^{-1}(0)^s) /\!/ G \subset \mathcal{M}_\chi \tag{13}$$

is smooth. Then $\mathcal{M}_\chi^s$ is symplectic and contains $T^*(X^s /\!/ G)$.