MA 822: NAKAJIMA QUIVER VARIETIES AND INSTANTON MODULI SPACES

Course name. MA 822—Topics in geometry.

Time and location. Tuesday and Thursday from 2:00 PM – 3:15 PM in CAS 314.

Summary.

A few decades ago Nakajima presented a beautiful geometric realization of a class of infinite-dimensional Lie algebras called Kac–Moody Lie algebras. Such Lie algebras act on the cohomology of certain geometric moduli spaces. The primordial instance of this is an action of an infinite-dimensional Lie algebra called the Heisenberg algebra which acts on the cohomology of the Hilbert scheme of points on a complex surface. More generally, these Kac–Moody Lie algebras act on moduli spaces of instantons which are solutions to the four-dimensional Yang–Mills equations.

This class begins with an overview of Hilbert schemes on complex surfaces. From here, we introduce more general geometric moduli spaces using the theory of quiver representations and quiver varieties which we will connect to moduli spaces of instantons. Every quiver has associated to it a Lie algebra of Kac–Moody type. We cover the requisite theory of cohomology and end with the geometric realization of Kac–Moody Lie algebras. Time permitting, we will also survey a contextualization of these constructions from the point of view of string theory (no knowledge of string theory is required).

Textbooks and recommended readings.

I will post lecture notes on the course website. In addition the following two textbooks will be useful.

- *Quiver representations and quiver varieties*, by Alexander Kirillov Jr.
- *Lectures on Hilbert schemes of points on surfaces*, by Hiraku Nakajima.

Prerequisites.

We will assume that the student has been exposed to the concept of a Lie algebra and a representation, some symplectic geometry, and basic algebraic geometry (such as what a variety and a scheme is). This is an extremely fast paced class and students are expected to attend every class. If you are falling behind with the material please contact me immediately.

Assessment.

The only assessment will be a written report on a topic related to the content of the class. This assessment is intended to simulate the process of writing a research paper. A short written proposal (no more than one page) for the topic must be submitted no later than **February 23**. The first draft of the report is due on **April 6**. The report must be written with your favorite TeX compiler and should be between

10–20 pages. More details on the expected form of the report will be announced in class. Following this, the report will be submitted to another student for peer review. Peer reviews must be returned back to the author by **April 20** for final revisions. The final report must be submitted by **May 2**.

Topics to be covered.

Here is a list of topics ordered roughly in the way that we will cover them in the course.

- Hilbert schemes on surfaces, framed torsion-free sheaves and their cohomologies.
- Quiver representations, quiver varieties, Kleinian singularities, and the geometric McKay correspondence.
- Hamiltonian reduction and GIT quotients.
- Instanton moduli space and the ADHM construction.
- The Heisenberg algebra and other Kac–Moody Lie algebras.
- Borel–Moore homology, convolution algebras, and a geometric realization of Kac–Moody Lie algebras.
- A physics realization of Nakajima's construction.