## November 2, 2022

We have many techniques to finding local extrema. Here is one that can be used to locate absolute extrema.

Suppose that $f(x)$ is defined on an interval and has exactly one local extrema at $x = c$.

- if it is a local maximum at $x = c$ then it is an absolute maximum.
- if it is a local minimum at $x = c$ then it is an absolute minimum.

Let's turn to an example that ties in a lot of the concepts we've discussed.

*Example* 2.43. Consider the polynomial $g(x) = x^2 - x + 1$. Using calculus, show that $g(x)$ has no (real) roots.

Notice that $\lim_{x\to\pm\infty} g(x) = \infty$. Next, we find the local extrema. We have

$$g'(x) = 2x - 1, \tag{92}$$

so there is a critical point at $x = 1/2$. But, $g''(1/2) > 0$, thus $g(x)$ attains a local minimum at $x = 1/2$. Since this is the only extrema, this is in fact an absolute minimum of the function.

Now, notice that

$$g(1/2) = 1/4 - 1/2 + 1 > 0. \tag{93}$$

Thus, $g(x)$ has no roots.

Here is a more involved example that uses similar ideas.

*Example* 2.44. Consider the function $f(x) = x \ln x - \ln x - x$ defined for $x > 0$. How many roots does the function $f(x)$ have? (Hint: use calculus.)

Notice that

$$\lim_{x\to 0^+} f(x) = \infty \tag{94}$$

and

$$\lim_{x\to\infty} f(x) = \infty. \tag{95}$$

Based on this, it is possible that $f$ could have zero, two, four, etc. roots. That is, we know it has an even number of roots. Let's nail this down.

Let's determine if the function has any local extrema. The derivative of $f(x)$ is

$$f'(x) = \ln x - \frac{1}{x}. \tag{96}$$

Notice that

$$\lim_{x\to 0^+} f'(x) = -\infty \tag{97}$$

and

$$\lim_{x\to\infty} f'(x) = \infty. \tag{98}$$

In particular we see that the derivative $f'(x)$ must have at least one zero.

Let's next compute the second derivative

$$f''(x) = \frac{1}{x} + \frac{1}{x^2}. \tag{99}$$

Notice that this function is always positive. Thus, $f'(x)$ is increasing everywhere. In particular, we see that there is a unique solution to $f'(x) = 0$, let's call it $x = c$. Sine $f''(c) > 0$ we see that $f$ has a local minimum (hence absolute minimum) at $x = c$.

Shuffling terms around, we see that $c$ satisfies

$$\ln(c) - \frac{1}{c} = 0. \tag{100}$$

Since $c > 0$, this is equivalent to $c \ln c - 1 = 0$. Thus

$$f(c) = c \ln c - \ln c - c = 1 - \frac{1}{c} - c = \frac{1}{c}\left(-c^2 + c - 1\right). \tag{101}$$

We showed in the last example that $-c^2 + c - 1 < 0$ for all $c$. Thus $f(c) < 0$.

We conclude that $f(x)$ has two roots.

We move on to optimization problems which generally ask the following:

- *What is the minimum or maximum value of a quantity subject to some class of constraints?*

Here is an example. Suppose we have a rectangle of lengths $x$ and $y$. Neither $x$ nor $y$ are fixed, but we do impose the constraint that

$$x + y = 1. \tag{102}$$

Then, for what values of $x, y$ is the area of the rectangle maximized and minimized?

To solve this, we recall that the area is

$$A = xy. \tag{103}$$

Notice that this is a function of two variables $x, y$. But, the constraint tells us how $x, y$ are related, namely $y = 1 - x$. Plugging this in, we see that the area is a function only of $x$

$$A(x) = x(1 - x) = x - x^2. \tag{104}$$

The critical points of $A(x)$ are ones where $A'(x) = 0$ which is the condition $1 - 2x = 0$. So there is only one critical point at $x = 1/2$. Since $A''(1/2) = -2$ this critical point is an absolute maximum. Notice that for $x = 1/2$ we have $y = 1/2$ by the constraint. This is a square, and the area is $A = 1/4$.

What about the absolute minimum. Naively, we would say that the function $A(x) = x - x^2$ has no absolute minimum. But we need to be careful since we are thinking of $x$ as a length, so we are only considering the function on the domain $x > 0$ and $x < 1$ (the last condition arises from the constraint and the fact that $y > 0$.) Thus, we still see that there is no absolute minimum (we can take $x$ to be closer and closer to zero or one and get a smaller and smaller area).

*Example* 2.45. Find the point on the curve $y = x^2$ that is closest to the point $(18, 0)$. (Hint: You may use the following fact

$$2 \cdot 2^3 + 2 - 18 = 0 \quad .) \tag{105}$$

Recall the formula for the distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$

$$d(P_1, P_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2. \tag{106}$$

A point on the curve $y = x^2$ is of the form $P_1 = (x, x^2)$. We want to minimize the distance to the point $P_2 = (18, 0)$. As a function of $x$, the distance squared is

$$D(x) = d(x)^2 = (x - 18)^2 + (x^2)^2 = (x - 18)^2 + x^4. \tag{107}$$

The derivative of this distance squared is

$$D'(x) = 2(x - 18) + 4x^3 = 2(2x^3 + x - 18). \tag{108}$$

We want to describe solutions to $D'(x) = 0$, so it suffices to look at the equation $2x^3 + x - 18 = 0$. One method is to observe that this can be factored like

$$(x - 2)(2x^2 + 4x + 9) = 0 \tag{109}$$

and hence the only solution is $x = 2$.

Thus the point $(2, 4)$ is the closest point on the curve of $y = x^2$ to the point $(18, 0)$.