We've introduced the definite integral

$$\int_a^b f(t)\,\mathrm{d}t \tag{166}$$

as the net area under between the graph $y = f(t)$ and the t-axis between $t = a$ to $t = b$. A precise definition was given in terms of a limit of Riemann sums.

Let's imagine fixing a function $f(t)$ and a starting point $t = a$ of some interval. Then, we can express the net area between the graph $y = f(t)$ and the t-axis between $t = a$ and some variable number $t = x$ as

$$N(x) = \int_a^x f(t)\,\mathrm{d}t. \tag{167}$$

Notice that N is a function of x and **not a function of** the "dummy" variable t.

Example 2.77. Find the area function $N(x)$ determined by the function $f(t) = t$ and the number $a = -1$. Is $N(x)$ a differentiable function? If so, compare the derivative A' to the original function f.

The example above is a simple illustration of a important theorem.

Theorem 2.78 (The fundamental theorem of calculus)**.** *Let* $N(x) = \int_a^x f(t)\mathrm{d}t$ *be the area function of the function* f. *Assume that* f *is continuous on the interval* $[a, b]$. *Then* $N(x)$ *is continuous on the interval* $[a, b]$ and *differentiable on the interval* (a, b). *Furthermore*

$$N'(x) = f(x) \tag{168}$$

on this interval. In particular, $N(x)$ *is an antiderivative for* $f(x)$ *on the interval* $[a, b]$.

Here is a sketch of the proof. Let's look at small sub interval $[x, x + h]$ where h is a small number. The area between the graph $y = f(t)$ on this interval is

$$N(x + h) - N(x). \tag{169}$$

If we assume that f is approximately constant on this interval, then this quantity is approximately the area of a rectangle

$$N(x + h) - N(x) \approx hf(x). \tag{170}$$

Or, dividing by h we have

$$\frac{N(x + h) - N(x)}{h} \approx f(x). \tag{171}$$

This line of reasoning can be turned into a proof that

$$N'(x) = \lim_{h \to 0} \frac{N(x + h) - N(x)}{h} = f(x). \tag{172}$$

Because $N(x)$ is an antiderivative for $f(x)$ we know that any other antiderivative $F(x)$ of $f(x)$ differs by a constant

$$F(x) = N(x) + C. \tag{173}$$

Of course, $N(a) = 0$, so that $C = F(a)$. In particular we obtain the following important corollary.

Corollary 2.79. *Suppose f is continuous on the interval $[a, b]$ and that $F(x)$ is an antiderivative for f on this interval. Then $N(b) = F(b) - F(a)$—equivalently, we have*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{174}$$

It will be useful to introduce the following notation

$$F(x)|_a^b = F(b) - F(a), \tag{175}$$

so that the above result can be written as

$$\int_a^b f(x)\mathrm{d}x = F(x)|_a^b. \tag{176}$$

Example 2.80. Evaluate

$$\int_0^3 e^{2x} \mathrm{d}x. \tag{177}$$

Example 2.81. Evaluate

$$\int_0^4 \frac{1}{4 + x^2} \mathrm{d}x. \tag{178}$$