NOVEMBER 30, 2022

Last time we reviewed the most important theorem in calculus. The fundamental theorem states that if f is a continuous function then

$$N'(x) = f(x) \tag{177}$$

where $N(x) = \int_a^x f(x)\mathrm{d}x$ is the net area function corresponding to f. Equivalently, this can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\int\limits_a^x f(x)\mathrm{d}x = f(x). \tag{178}$$

As a consequence, $N(x) = \int_a^x f(x)\mathrm{d}x$ is an antiderivative of the function f. From this we deduce that if $F = \int f(x)\mathrm{d}x$ (this is the indefinite integral symbol) is any other antiderivative of f then

$$\int\limits_a^b f(x)\mathrm{d}x = F(b) - F(a) \stackrel{\text{def}}{=} F(x)|_a^b. \tag{179}$$

Example 2.80. Evaluate

$$\int\limits_0^3 e^{2x}\mathrm{d}x. \tag{180}$$

Example 2.81. Evaluate

$$\int\limits_0^1 \frac{1}{3+3x^2}\mathrm{d}x. \tag{181}$$

It's a good time to study some properties of the definite integral.

• For any continuous function f and a any number, one has

$$\int_a^a f(x)\mathrm{d}x = 0. \tag{182}$$

This is because the net area is clearly zero as the width of the interval is null.

• For any continuous function f and numbers $a < b < c$ one has

$$\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x. \tag{183}$$

Geometrically this is just partitioning the region of integration into two smaller regions.

One thing we've sort of taken for granted is that $\int_a^b f(x)\mathrm{d}x$ is defined when $a < b$ (using the definition in terms of net area). It's easy to extend this more generally.

• For any continuous function f an numbers $a < b$ one has

$$\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x. \tag{184}$$

One can take this as a definition.

• For any two functions f, g one has

$$\int_a^b (f(x) + g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x. \tag{185}$$

• For any constant c one has

$$\int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x. \tag{186}$$

Example 2.82. Compute

$$\int_1^2 (x^2 + 3x - 1)\mathrm{d}x \tag{187}$$

Let's give a formal proof of the fundamental theorem. Let $N(x) \int_a^x f(x)\mathrm{d}x$ be the net area function, then its derivative, by definition, is

$$N'(x) = \lim_{h\to 0} \frac{N(x+h) - N(x)}{h}. \tag{188}$$

Using formal properties of integrals we have the following relation

$$N(x+h) - N(x) = \int_a^{x+h} f(x)\mathrm{d}x - \int_a^{x} f(x)\mathrm{d}x = \int_x^{x+h} f(x)\mathrm{d}x. \tag{189}$$

Let $m(x), M(x)$ be the minimum and maximum values of f on the small interval $[x, x+h]$. (These exist by the extreme value theorem.) Then

$$m(x)\cdot h \leq \int_x^{x+h} f(x)\mathrm{d}x \leq M(x)\cdot h. \tag{190}$$

Or equivalently using (188) we obtain

$$m(x) \leq \frac{N(x+h) - N(x)}{h} \leq M(x). \tag{191}$$

By the squeeze theorem we can take the limit of this expression as $h \to 0$. Indeed, since f is continuous, both $m(x), M(x)$ approach $f(x)$ as $h \to 0$. In particular, the derivative of $N(x)$ exists and $N'(x) = f(x)$ as desired.

Example 2.83. Simplify the following expression

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_1^{x^3} e^{x^2}\,\mathrm{d}x. \tag{192}$$

(Use the fundamental theorem, do not try to find the antiderivative.)