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Another optimization problem.

Example 2.46. A fence that is 6 feet high is running parallel to a house and is 3 feet away from the house. What is the minimum length of a ladder which clears the fence and touches the house?

Let's denote by x the distance, in feet, from the base of the ladder to the fence. Then the length between the base of the ladder and the house is $x+3$. Let y be the height from the top of the ladder to the ground. Then, the length of the ladder ℓ can be computed as

$$\ell^2 = (x+3)^2 + y^2. \tag{110}$$

The quantities x, y are related by the following geometric constraint. Using similar triangles we see that we have

$$\frac{y}{x+3} = \frac{6}{x}. \tag{111}$$

In other words, $y = \frac{6(x+3)}{x}$.

We can use this to express the length of the ladder just in terms of the quantity x

$$\ell(x)^2 = (x+3)^2 + \frac{36}{x^2}(x+3)^2 = (x+3)^2\left(1+\frac{36}{x^2}\right). \tag{112}$$

We want to minimize the function $L(x) = \ell(x)^2$.

The first derivative is

$$\begin{aligned} L'(x) &= 2(x+3)\left(1+\frac{36}{x^2}\right) + (x+3)^2\left(-\frac{72}{x^3}\right) \\ &= (x+3)\left(2(1+36/x^2) - (x+3)(72/x^3)\right) \\ &= (x+3)\left(2+72/x^2 - 72/x^3 - 216/x^3\right) \\ &= (x+3)\left(2 - 216/x^3\right). \end{aligned}$$

We have potential critical points at $x = -3$ and $x = \sqrt[3]{108} = 3\sqrt[3]{4}$. The point $x = -3$ is not meaningful since $x > 0$ represents a length. Therefore we have only one critical point!

By computing $L''(3\sqrt[3]{4}) > 0$ (or using the first derivative test) we see that this is a local, hence absolute, minimum.