We continue with L'Hôpital's rule. Recall that the first indeterminate form '0/0' that we can apply this rule to can be carefully stated as follows.

Theorem 2.50. *Suppose that f, g are differentiable on an interval containing $x = a$ and assume that $g'(x) \neq 0$ on this interval for $x \neq a$. Also, assume that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{129}$$

Example 2.51. Evaluate the limit if it exists

$$\lim_{x\to 0} \frac{e^x - 1}{\sqrt{x^3+1} - 1}. \tag{130}$$

Let's now consider the indeterminate form ∞/∞.

Theorem 2.52. *Suppose that f, g are differentiable on an open interval containing a with $g'(x) \neq 0$ for $x \neq a$ on this interval. If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{131}$$

Notice that this is the same conclusion as in the other indeterminate form $0/0$.

We can use this to alternatively compute some familiar limits.

Example 2.53. Compute the limit using L'Hôpital's rule

$$\lim_{x\to\infty} \frac{x^3+3}{2x^3+x^2+1} \tag{132}$$

Let's now move onto an example that isn't immediately in one of the indeterminate forms above.

Example 2.54. Compute the limit if it exists

$$\lim_{x\to 0^+} x \cdot \ln x. \tag{133}$$

Notice that naively plugging in 0 (or really a small positive number) we will get the indeterminate form $0 \cdot (-\infty)$. To do this example we rewrite the function as

$$x \ln x = \frac{\ln x}{1/x}. \tag{134}$$

Note that now we have put this in the indeterminate form $-\infty/\infty$, so we can apply the rule.

$$\lim_{x\to 0^+} x \cdot \ln x = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = -\lim_{x\to 0^+} x = 0. \tag{135}$$

Example 2.55. Evaluate

$$\lim_{x\to\infty} \frac{\ln x}{x}. \tag{136}$$

The above limits suggest insight into the 'growth rate' of functions. We say that a function g grows faster than a function f if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0. \tag{137}$$

Equivalently, $\lim_{x\to\infty} g(x)/f(x) = \infty$. We say that the growth rates of f, g are comparable if the limit $\lim_{x\to\infty} f(x)/g(x)$ is a finite nonzero number.

Example 2.56. In the last example, we saw that x grows faster than $\ln x$.

Example 2.57.

- Are there any $p, q > 0$ such that $(\ln x)^p$ and x^q have comparable growth?
- Are there any $p, q > 0$ such that e^{px} and x^q have comparable growth?