OCTOBER 11, 2022

Let's begin with one more example related to implicit derivatives.

Example 2.24. Find equations for lines tangent to the graph

$$y^2 - 3xy = 2 \tag{69}$$

when $x = 1/3$.

Next, we will move onto derivatives of logarithms. The natural logarithm $\ln x$ is, by definition, the inverse to the exponential function e^x. Recall that two functions $f(x), g(x)$ are said to be inverses of one another if

$$f(g(x)) = x, \quad \text{and} \quad g(f(x)) = x. \tag{70}$$

Thus, the defining properties of the natural logarithm are

$$e^{\ln x} = x, \quad \text{and} \quad \ln e^x = x. \tag{71}$$

Some key identities to keep in mind when working with logarithms include $\ln(ab) = \ln a + \ln b$ and $\ln(a^r) = r \ln a$.

Just the knowledge of the properties in (71) will allow us to nail down the derivative of $\ln x$ using the rules that we know. First, introduce a dependent variable $y = \ln x$ into the first expression above:

$$e^y = x. \tag{72}$$

Next, we take the implicit derivative with respect to x to get

$$e^y y' = 1. \tag{73}$$

In other words, $y' = e^{-y}$. Substituting $y = \ln x$ we then obtain an explicit expression for the derivative.

Proposition 2.25 (Derivative of natural log)**.**

$$(\ln x)' = e^{-\ln x} = \frac{1}{x}. \tag{74}$$

Example 2.26. Use the second expression in (71) to 'rederive' the formula $(e^x)' = e^x$.

The expression $\ln x$ is only defined for $x > 0$. On the other hand, the expression $\ln|x|$ is defined for any $x \neq 0$.

Example 2.27. Use the chain rule to show that for all $x \neq 0$ one has

$$(\ln|x|)' = \frac{1}{x}. \tag{75}$$

Example 2.28. Compute the derivative of $\ln\left(\sqrt{x+1}\right)$ in two ways.

A simple application of chain rule shows the following.

Proposition 2.29. *Suppose that f is differentiable and $\ln(f(x))$ exists. Then*

$$(\ln(f(x)))' = \frac{f'(x)}{f(x)}. \tag{76}$$

Next, we consider functions which are like exponentials but where we use a different base. For example, consider the function $f(x) = 2^x$. To compute the derivative we introduce the dependent variable $y = 2^x$ and write this expression as

$$y = 2^x \leftrightarrow \ln y = \ln(2^x) = x\ln 2. \tag{77}$$

Thus, the original equation $y = 2^x$ is equivalent to the implicit representation

$$\ln y = x\ln 2. \tag{78}$$

We then apply the derivative to get

$$\frac{y'}{y} = \ln 2. \tag{79}$$

In other words $y' = y\ln 2 = 2^x\ln 2$, so that

$$(2^x)' = 2^x\ln 2. \tag{80}$$

Without much more difficulty one can show.

Proposition 2.30. *For $0 < b$ and $b \neq 1$ one has*

$$(b^x)' = b^x\ln b. \tag{81}$$

Example 2.31. Find the slope of the line tangent to the graph of $f(x) = x^x$ at $x = 1$. Does this graph have any horizontal tangent lines?