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Last time we defined the notions of absolute minima and absolute maxima.

Example 2.32*.* Let's start with some simple examples.

- Does the function $f(x) = \arctan(x)$ defined over the real numbers have any extreme values?
- Does the function $f(x) = \sqrt{x}$ defined on the interval $[0, 1]$ have any extreme values? What about on the interval $(0, 1)$?

An important theorem about absolute extrema is the following.

Theorem 2.33 (Extreme value theorem)**.** *Suppose a function $f(x)$ is defined and continuous on the closed interval $[a, b]$. Then $f(x)$ has an absolute minimum and absolute maximum.*

We now turn to *local* extrema.

Definition 2.34. Suppose c is an interior point of some interval I on which a function f is defined. If $f(c) \geq f(x)$ for all x in I then $f(c)$ is a local maximum value of f. If $f(c) \leq f(x)$ for all x in I then $f(c)$ is a local minimum value of f. In either case we call $f(c)$ a local extreme value.

In each case we say that f has a local extreme value at the point c.

Warning: In this class we adopt the convention that local extrema can occur only at interior points of the interval.

Example 2.35. By analyzing the graph, locate all local and absolute extrema of the function $f(x) = x^4 - 2x^2 + 1$.

Calculus comes in as an efficient tool to locate local extrema.

Theorem 2.36. *If f has a local extreme value at a point c then $f'(c) = 0$.*

The converse statement, namely that $f'(c) = 0$ implies that $f(c)$ is a local extrema is false.

Example 2.37. Does the function $f(x) = x^3$ have any local extrema?

Also, the claim that extreme values only arise at points where $f'(c) = 0$ is *false*.

Example 2.38. Locate all local extrema of the function $f(x) = |x|$.

Definition 2.39. A *critical point* c in the domain of f is one for which either:

- $f'(c) = 0$ or
- $f'(c)$ fails to exist.

Example 2.40. Locate all local and absolute extrema of the function $f(x) = x^3 \ln x$.

Example 2.41. A ball is launched into the air from a cliff that is 100 ft above the ground at a speed of 25 ft/s. Its height above the ground at time t is given by

$$h(t) = -16t^2 + 25t + 100. \tag{82}$$

What is the maximum height that the ball reaches?