SEPTEMBER 16, 2022

Today we will review material in sections 2.4–2.5 of the book.

Recall that we say

$$\lim_{x\to a} f(x) = \infty \tag{1}$$

if we can make the value $f(x)$ be infinitely large by choosing x to be sufficiently close to a. Similarly, we can define

$$\lim_{x\to a} f(x) = \pm\infty, \quad \lim_{x\to a^{\pm}} f(x) = \pm\infty. \tag{2}$$

In any of these cases we call the line $x = a$ a vertical asymptote of the function f.

0.0.1. Challenge question. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = -\infty$ then what can we say about $\lim_{x\to a}(f(x) + g(x))$?

Example 0.1. Compute the following limit (if it exists)

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2}\right) \tag{3}$$

Example 0.2. Compute the following limit (if it exists)

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{2x+1}{x}\right). \tag{4}$$

Example 0.3. Define the function

$$f(x) = \frac{x^2 - x - 2}{x - a} \tag{5}$$

where a is some number. For which values of a do the following limits exist.

- $\lim_{x \to a^+} f(x)$.
- $\lim_{x \to a^-} f(x)$.
- $\lim_{x \to a} f(x)$.

0.0.2. Recall that we say

$$\lim_{x\to\infty} f(x) = L \tag{6}$$

if we can make $f(x)$ as close to the value L for x a large enough number. In this case we say that $y = L$ is a horizontal asymptote of the function f.

Example 0.4. Find the limit (if it exists)

$$\lim_{x\to\infty} \frac{x^2 + 2x + 1}{4x^2 + 5} \tag{7}$$

Example 0.5. Find the limit (if it exists).

$$\lim_{x\to\infty} \frac{e^x \sin x}{e^{2x} + 1} \tag{8}$$

Definition 0.6. Suppose that a function f is defined on an open subset of $\mathbf{R}$ which contains the point a. We say that f is continuous at a if

$$f(a) = \lim_{x \to a} f(x). \tag{9}$$

0.0.3. Examples and non-examples of continuous functions .

0.0.4. Checklist for continuity. For a function f to be continuous at a point a, the following three conditions must hold.

(1) f must be defined at a.
(2) the limit $L = \lim_{x \to a} f(x)$ must exist.
(3) the limit L must equal $f(a)$.

Example 0.7. At which points $a \in \mathbf{R}$ is the function $f(x) = x/|x|$ continuous?

0.0.5. Properties of continuous functions. We say a function is continuous if it is continuous at all points in its domain.

(1) the sum, product, difference, and ratio (when defined) of two continuous functions is again continuous.
(2) all polynomials are continuous.
(3) Rational functions (ratios of polynomials) are continuous at every point in their domain.
(4) the composition of two continuous functions is again continuous.

0.0.6. Continuity on closed intervals. So far we've covered what it means for a function to be continuous on domains which are open intervals, like $(0, \infty)$. What about the function $f(x) = \sqrt{x}$ which is defined on the closed interval $[0, \infty)$? The problem is at the closed point $0 \in [0, \infty)$. In addition to checking continuity on the open interval $(0, \infty)$ we also need to make sure that the right sided limit at 0 agrees with value of the function at that point

$$\lim_{x \to 0^+} f(x) \stackrel{?}{=} f(0). \tag{10}$$

If this is the case then we say that f is continuous on the closed interval $[0, \infty)$.

September 22, 2022

In the first week we defined the notion of average velocity which is the average rate of change of quantity position. We then motivated the definition of a limit by the idea of instantaneous rate of change. In this lecture we return to the precise definition of the instantaneous rate of change of a function at a point—this is called the *derivative* of the function at a point.

Definition 0.8. The *derivative* of a function f at a point a is the limit

$$f'(a) \stackrel{\text{def}}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

when it exists.

We say that f is *differentiable* at $x = a$ if the derivative at a exists. Otherwise we say that f is not differentiable at a.

Example 0.9. Using the limit definition above to compute the derivative of the function $f(x) = x^2 - 3x$ at the point $x = 1$.

Graphically, the average rate of change of a function $y = f(x)$ on the interval $(a, a + \Delta x)$ is defined as

$$\frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{f(a + \Delta x) - f(a)}{(a + \Delta x) - \Delta x} = \frac{f(a + \Delta x) - f(a)}{\Delta x} \tag{11}$$

We understood instantaneous rate of change of a function $y = f(x)$ as the limit

$$\frac{\Delta y}{\Delta x} \xrightarrow{\Delta x \to 0} f'(a) \tag{12}$$

Thus, the derivative at $x = a$ measures the *slope* of the graph at that point $(a, f(a))$.