Today we will cover the product and quotient rules for computing derivatives.

Remember, the derivative $f' = \mathrm{d}f/\mathrm{d}x$ is motived by the concept of a 'rate of change'. For average rate of change we wanted to consider how a function $f(x)$ changed as we change x by some amount Δx:

$$x \rightsquigarrow x + \Delta x. \tag{29}$$

When we change x like this, the function $f(x)$ will then change like

$$f(x) \rightsquigarrow f(x + \Delta x) = f(x) + \Delta f(x) \tag{30}$$

where $\Delta f(x) = f(x + \Delta x) - f(x)$.

We now want to consider what happens when we have a product of two functions $f \cdot g$. Then, we can think of $f \cdot g$ as changing like

$$(f + \Delta f) \cdot (g + \Delta g) = f \cdot g + \Delta f \cdot g + f \cdot \Delta g + (\Delta f) \cdot (\Delta g). \tag{31}$$

In calculus we study the instantaneous rate of change, which is the limit of the rate of change as the interval gets infinitesimally small. So, we can imagine that the quantities Δf and Δg are extremely small. Then, if this is the case, we can think of the quantities $f \cdot \Delta g$ and $\Delta f \cdot g$ as being quite large as compared to $\Delta f \cdot \Delta g$.

In other words, we see that when Δf and Δg are very small, the amount that $f \cdot g$ changes by is approximately

$$\Delta f \cdot g + f \cdot \Delta g.$$

This reasoning can be turned into the following theorem.

Theorem 2.1 (Product rule)**.** *Suppose that f, g are differentiable functions. Then*

$$(f \cdot g)' = f' \cdot g + f \cdot g'. \tag{32}$$

Example 2.2. Using the power rule, "check" the product rule for the product of the two functions $f(x) = x^3$ and $g(x) = 5x^7$.

Example 2.3. Compute the derivative of the function $f(x) = x^2 e^x$.

If f, g are two differentiable functions and g is nonzero (in some domain), then we can consider the derivative of the function $h = f/g$. Notice that we could also write this equation like $f = g \cdot h$. Applying the product rule we see that

$$f' = g' \cdot h + g \cdot h'. \tag{33}$$

But, remember that we really wanted to know what h' is. For this, we can use the above equation to solve for this

$$h' = \frac{f' - g' \cdot h}{g} = \frac{f' - g' \cdot f/h}{g}. \tag{34}$$

Rewriting this equation (in a completely equivalent way) results in the usual formulation of the quotient rule.

Theorem 2.4 (Quotient rule)**.** *If f, g are differentiable functions and f/g is defined, then*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}. \tag{35}$$

Example 2.5. Where is the following function differentiable?

$$f(x) = \frac{xe^x}{1 + x^2}. \tag{36}$$

Compute the derivative of the function when it is defined.

Example 2.6. Compute the derivative of

$$f(x) = \frac{1 - x^3}{1 - x}. \tag{37}$$

Find the equation of the line tangent to the graph of $f(x)$ at $x = -1$.