Today we will discuss derivatives of trigonometric functions like

$$\sin(x), \quad \cos(x), \quad \tan(x), \quad \text{etc.} \tag{38}$$

Recall the following limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{39}$$

One nice proof of this uses some basic trigonometry for expressions of areas of triangles.

Example 2.7. What is the derivative of the function $\sin x$ at $x = 0$?

Another nice limit is the following

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0. \tag{40}$$

To obtain this notice that for $x \neq \pi, 3\pi, \ldots$ one has

$$\frac{\cos x - 1}{x} = \frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)}. \tag{41}$$

Now $\sin^2 x + \cos^2 x = 1$, so we can write this expression as

$$-\frac{\sin^2 x}{x(\cos x + 1)} = -\frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}. \tag{42}$$

Using the fact that the limit of the product of two functions is the product of the limits, we see that as $x \to 0$ this limit is $1 \cdot 0 = 0$.

Example 2.8. What is the derivative of the function $\cos x$ at $x = 0$?

Let's move on to general formulas for the derivatives of the functions $\sin x$ and $\cos x$.

Proposition 2.9. *One has*

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x. \tag{43}$$

Proof. Let's consider just the derivative of $\sin x$. We appeal to a (hopefully familiar) trigonometric identity

$$\sin(a+b) = \sin a \cos b + \cos a \sin b. \tag{44}$$

Now we compute

$$\begin{aligned}
(\sin x)' &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \sin x \cdot \left(\lim_{h\to 0} \frac{\cos h - 1}{h}\right) + \cos x \cdot \left(\lim_{h\to 0} \frac{\sin h}{h}\right) \\
&= \sin x \cdot 0 + \cos x \cdot 1.
\end{aligned}$$

□

Example 2.10. Compute the derivative of the function

$$f(x) = e^x \cos(x) \sin(x). \tag{45}$$

There are other trigonometric functions we can build using sine and cosine. For example, the tangent and cotangent functions are

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot(x) = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \tag{46}$$

And the secant and cosecant functions are

$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}. \tag{47}$$

Example 2.11. When is the function $\tan x$ well-defined? Use the quotient rule to compute the derivative

$$(\tan x)'. \tag{48}$$

Use this to find the equation of the line tangent to the graph $y = \tan x$ at the point $x = \pi/4$.

Let's return to a real-world application of derivatives. We've discussed that if $s(t)$ is the position of a particle as a function of time then the derivative $s'(t)$ represents the velocity as a function of time:

$$s'(t): \quad \text{velocity at time } t. \tag{49}$$

In other words, the rate of change of position is velocity.

As we say last time, we can iterate the process of taking the derivative. In other words, we can consider the second derivative $s''(t)$ This represents the rate of change of velocity as a function of time and is called the *acceleration*.

Example 2.12. Suppose a ball was thrown off of a cliff into the water at time $t = 0$ and its height (in meters) above sea level (before hitting the water) is described by the following function if time (in seconds):

$$s(t) = 35 + 11t - 10t^2. \tag{50}$$

At what time does the ball hit the water? At what time is the ball moving the fastest? Describe the acceleration of the ball.