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HIGHER-DIMENSIONAL CHIRAL ALGEBRAS IN THE JOUANOLOU MODEL

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Abstract. We appeal to the theory of Jouanolou torsors to model the coherent cohomology of configuration spaces of points in affine space $\mathbb{A}^d$. Using this model, we develop the operadic notion of chiral operations, thus generalizing the notion of *chiral algebras* of [2] to higher dimensions. To produce examples, we use a higher-dimensional conceptualization of the residue which is inspired by Feynman graph integrals.

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Introduction

The concept of a *chiral algebra* was introduced by Beilinson and Drinfeld as a geometrization of the so-called *operator product expansion* which describes the algebraic structure of local operators in (real) two-dimensional chiral conformal field theory (CFT) [2]. The primary goal of this paper is to introduce a model for the algebraic structure behind the (derived) operator product expansion in higher dimensional *holomorphic* quantum field theory.

For a chiral algebra on an algebraic variety X, the chiral operations are parametrized by configuration spaces of points on X. For example, on $\mathbb{A}^1$ all such configuration spaces are affine, so algebraically it suffices to pass to the algebra of holomorphic functions on configuration spaces. On the other hand, for the configuration spaces of points in $\mathbb{A}^d, d > 1$ are no longer affine so to formulate a higher-dimensional theory of chiral operations it is essential that we instead pass to the derived global sections.

In complex geometry, there is a strict commutative dg algebra which models derived global sections given by the Dolbeault complex. What we require is an analog of the Dolbeault complex that can be used in the algebraic setting. Following the ideas of [5], we use the so-called Jouanolou construction to accomplish this. We first find an affine variety, affinely fibered over the configuration space $\mathrm{Conf}_n(\mathbb{A}^d)$. The Jouanolou trick is to then consider the relative de Rham complex of this fibration as the explicit derived model for the configuration space. When the number of points is $n = 2$ we have $\mathrm{Conf}_2(\mathbb{A}^d) = (\mathbb{A}^d - \{0\}) \times \mathbb{A}^d$ and this model is used in [5] to give a definition of the dg Lie algebra of derived currents valued in a Lie algebra (so, it plays the role of the Laurent series when $d = 1$).

Chiral algebras are defined within the context of $\mathcal{D}$-modules. For $\mathcal{D}$-module $\mathcal{A}$ defined on an algebraic curve X, the space of n-ary chiral operations, following [2], is

$$\mathrm{Hom}_{\mathcal{D}_{X^n}}\left(j_* j^* \mathcal{A}^{\boxtimes n}, \Delta_* \mathcal{A}\right) \tag{0.1}$$

where $j\colon \mathrm{Conf}_n(X) \to X \times \cdots \times X$ is the open immersion and $\Delta\colon X \to X \times \cdots \times X$ is the diagonal embedding. Combining chiral operations of all arities results in an operad, and a chiral algebra is simply a Lie algebra object in this operad. In genealizing this to $X = \mathbb{A}^d$ we use the Jouanolou trick to provide an explicit model for the $\mathcal{D}$-module derived pushforward $\mathbf{R} j_* j^* \mathcal{A}^{\boxtimes n}$.

Let $\mathcal{A}$ be an $\mathcal{D}$-module on $\mathbb{A}^d$. Then, for any finite set I, the Jouanolou model $\mathbf{J}^I_{\mathbb{A}^d}[\mathcal{A}]$ (see definition 1.4) is a model for the $\mathcal{D}$-module derived pushforward $\mathbf{R} j_* j^* \mathcal{A}^{\boxtimes I}$. Furthermore, as the set I runs over finite sets, the collection

$$\left\{\mathrm{Hom}_{\mathcal{D}_{(\mathbb{A}^d)^I}}\left(\mathbf{J}^I_{\mathbb{A}^d}[\mathcal{A}]\ ,\ \Delta^I_* \mathcal{A}\right)\right\}_I \tag{0.2}$$

forms a dg operad. Denote this dg operad by $\mathcal{P}_d$. Notice that by construction this operad is GL_d-equivariant. This is the main advantage of the Jouanolou model as compared to the Čech–Thom–Sullivan cosimplicial model for configuration spaces used in [6].

Definition. A d-dimensional ***homotopy chiral algebra***, in the Jouanolou model, is map of dg operads

$$\mathcal{L}\mathrm{ie}_\infty \to \mathcal{P}_d[\mathcal{A}] \tag{0.3}$$

where $\mathcal{L}\mathrm{ie}_\infty$ is the dg operad of L_∞ algebras.

Homotopy chiral algebras in the Jouanolou model define chiral algebras in the sense of [7], but we make no assertion that the resulting ∞-category of Jouanolou chiral algebras is equivalent to the ∞-categorical description of [7].

The next main goal of this article is to provide explicit descriptions and examples of chiral algebras in higher dimensions. Firstly, we describe, in explicit detail, the chiral operations underlying the *unit* chiral algebra on $\mathbb{A}^d$. As is familiar from the theory of vertex algebras, the Cauchy residue encodes the chiral operation of the unit chiral algebra when $d = 1$. We generlize the residue to higher-dimensional configuration spaces in the Jouanolou model. The key to our definition of the residue is the use of Feynman graph integrals in higher-dimensional holomorphic quantum field theory. The analysis which underlies the convergence of such graph integrals uses the compactification of Schwinger spaces following [19]. A peculiar fact is that in dimensions $d > 1$ the unit chiral algebra (in the Jouanolou model) has nontrivial chiral operations of arity k for all $k = 1, 2, 3, \ldots$.

The next part of this paper explores further examples of higher-dimensional chiral algebras in the Jouanolou model. Using the higher-dimensional residue, we formulate a higher-dimensional version of Wick's theorem for chiral algebras in section 3 This leads to explicit descriptions of so-called *free* chiral algebras (those which come from free holomorphic quantum field theories). As an application, we provide a higher-dimensional free-field realization of the higher-dimensional Kac–Moody algebra of [5].

In the last part of the paper we focus on the case of complex dimension $d = 2$. We flip much of the logic of the bulk of this paper in reverse, by using the L_∞ relations to explicitly characterize Feynman graph integrals. As we have stated, holomorphic quantum field theory provides our inspiration for the definition of the higher-dimensional residue in the Jouanolou model. Explicitly, our residues are given as sums over weights of Feynman graph integrals. We show in section 2 that these residues satisfy an appropriate L_∞-relation, this is the essential structure underlying the unit chiral algebra in higher dimensions. In section 4, we utilize the L_∞-relations to formulate a *recursive* formula for weights of Feynman graph integrals in complex dimension two. The formula holds for graphs of a special type, which we refer to as Laman type I'.[1] We can state the recursive formula, heuristically, as follows.

Theorem (see theorem 4.13 for the precise formulation)**.** *Suppose the complex dimension is* $d = 2$*, and suppose that a chiral algebra has two-ary operation expressed in terms of the residue as we have defined it. Then, the chiral operation associated to an arbitrary graph* Γ *of type Laman I' is*

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[1] The term Laman graph is derived from [15].

completely fixed by the L_∞ relations. Namely, there is a recursive formula for the chiral operation associated to Γ constructed using the residue.

As an application of this recursive formula, we explore a class of graph integrals when $d = 2$. We compute, in full generality and explicitly, the weight of Laman I' Feynman graphs at loop orders one, two, and three (the latter using `Mathematica`). At lowest order in derivatives, we find an exact agreement with the work of physicists in [8]. For example, in the case of the two-loop Θ-diagram, in complex dimension $d = 2$, we find the chiral operation

$$\mu_4(P_{30}P_{32}P_{10}P_{12}P_{20}d\mathbf{z}_{0,1,2,3}) = \frac{1}{24}\left(\lambda_1 \wedge (\lambda_3 + 2\lambda_2)\right) \cdot \left(\lambda_3 \wedge (\lambda_1 + 2\lambda_2)\right) \cdot d\mathbf{z}_0^{\blacklozenge}, \tag{0.4}$$

to lowest order in derivatives. We introduce the notations in the main part of the text. (for example, $d\mathbf{z}^{\blacklozenge}$ denotes the holomorphic volume form shifted in cohomological degree, and P denotes the propagator). Our compution is logically independent and more general than [4] in the sense that we also include arbitrary derivative insertions at vertices.

Conventions.

- Let $\mathbf{k}$ be a field with $\mathrm{char}(\mathbf{k}) = 0$. For an affine variety $X = \mathrm{Spec}(\mathbf{k}[X])$ over $\mathbf{k}$ and a quasi-coherent sheaf M on X, we will not distinguish M itself and $\Gamma(X, M)$ the space of global sections. We will mainly work over the complex numbers $\mathbf{k} = \mathbb{C}$, but many definitions in this paper works for general field $\mathbf{k}$.
- We will denote by **fSet** the category of non-empty finite sets with surjective morphisms.
- For $I = \{1, \dots, k\} \in \mathbf{fSet}$ and a variety X, we write X^I for $X_1 \times \cdots \times X_k$ where X_i is a copy of X. Suppose that M is a quasi-coherent sheaf over X, we will denote $M^{\boxtimes I}$ the exterior product $M_1 \boxtimes \cdots \boxtimes M_k$ over X^I where M_i over $X_i = X$ is the sheaf M.
- A graded vector space V is a direct sum of vector spaces $V = \bigoplus_{i\in\mathbb{Z}} V_i$ and we say that V_i is in degree i. We will use the k-th shift notation $V[k], k \in \mathbb{Z}$ as well as the suspension notation $s^{-k}V, k \in \mathbb{Z}$

$$(V[k])_i = V_{i+k}.$$

  We also use $|a|$ for the degree of a homogeneous element $a \in V$.
- Given a manifold M, we use $\mathcal{A}^\bullet(M)$ to denote the space of complex-valued differential forms on M. If M is a complex manifold, to emphasize their bi-graded nature, we will use $\mathcal{A}^{\bullet,\bullet}(M)$ instead.

- We use the following Koszul sign rules for integrals: If M, N are manifolds whose dimensions are m, n respectively. Let $\alpha \in \mathcal{A}^i(M), \beta \in \mathcal{A}^j(N)$, then
$$\int_M \int_N \alpha \wedge \beta = (-1)^{ni} \int_M \alpha \int_N \beta.$$

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1. THE JOUANOLOU MODEL AND A DEFINITION OF HIGHER CHIRAL ALGEBRA

1.1. **The Jouanolou model for configuration space.** The reference is [2, Section 4.1.3, pp279]. Suppose $\pi : Y \to X$ is a Juanolou map; i.e., Y is a torsor over X with respect to an action of some vector bundle such that Y is an affine scheme. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, the global relative de Rham complex
$$\Gamma(Y, \Omega_{Y/X} \otimes \pi^*\mathcal{F})$$
is a model for $\mathbf{R}\Gamma(X, \mathcal{F})$. In fact, we have quasi-isomorphisms
$$\Gamma(Y, \Omega_{Y/X} \otimes \pi^*\mathcal{F}) \sim \mathbf{R}\Gamma(Y, \Omega_{Y/X} \otimes \pi^*\mathcal{F}) \sim \mathbf{R}\Gamma(X, \mathbf{R}\pi_*(\Omega_{Y/X} \otimes \pi^*\mathcal{F})).$$

Because π is a Zariski locally trivial fibration with fiber $\mathbb{A}^N$, the Poincare lemma for differenitial forms on $\mathbb{A}^N$ implies that the inclusions
$$\mathcal{F} \hookrightarrow \pi_*(\Omega_{Y/X} \otimes \pi^*\mathcal{F}) \hookrightarrow \mathbf{R}\pi_*(\Omega_{Y/X} \otimes \pi^*\mathcal{F})$$
are quasi-isomorphisms.

For the finite set $I = \{1, \dots, k\}$, we define a Jouanolou torsor over $\mathrm{Conf}_I(\mathbb{A}^d)$
$$\mathbb{J}^I_{\mathbb{A}^d} := \{\sum_{s=1}^{d} x^s_{ij}(z^s_i - z^s_j) = 1 | 1 \le i < j \le k\} \subset (\mathbb{A}^d)^k \times (\mathbb{A}^d)^{\binom{k}{2}}$$
with the Jouanolou map
$$p_I \colon \mathbb{J}^I_{\mathbb{A}^d} \to \mathrm{Conf}_I(\mathbb{A}^d)$$
being the obvious projection. The projection p_I is affine with fiber isomorphic to $\mathbb{A}^{\binom{k}{2}}$ and carries an action of the vector bundle $\mathbb{V}^I_{\mathbb{A}^d} \to \mathrm{Conf}_I(\mathbb{A}^d)$ defined by
$$\mathbb{V}^I_{\mathbb{A}^d} := \{\sum_{s=1}^{d} v^s_{ij}(z^s_i - z^s_j) = 0 | 1 \le i < j \le j\} \subset (\mathbb{A}^d)^k \times (\mathbb{A}^d)^{\binom{k}{2}}.$$

Lemma 1.1. *We have* $\Omega^\bullet_{\mathbb{J}^I_{\mathbb{A}^d}/\mathrm{Conf}_I(\mathbb{A}^d)} = \Omega^\bullet_{\mathbb{J}^I_{\mathbb{A}^d}/(\mathbb{A}^d)^I}$.

Proof. This follows from a standard fact in algebraic geometry. Since $j_I\colon \mathrm{Conf}_I(\mathbb{A}^d) \hookrightarrow (\mathbb{A}^d)^I$ is an open immersion, the sheaf of relative differentials $\Omega^1_{\mathrm{Conf}_I(\mathbb{A}^d)/(\mathbb{A}^d)^I} = 0$. The sequence of maps $\mathbb{J}^I_{\mathbb{A}^d} \xrightarrow{p_I} \mathrm{Conf}_I(\mathbb{A}^d) \xrightarrow{j_I} (\mathbb{A}^d)^I$ induces an exact sequence

$$p_I^*\Omega^1_{\mathrm{Conf}_I(\mathbb{A}^d)/(\mathbb{A}^d)^I} \to \Omega^1_{\mathbb{J}^I_{\mathbb{A}^d}/(\mathbb{A}^d)^I} \to \Omega^1_{\mathbb{J}^I_{\mathbb{A}^d}/\mathrm{Conf}_I(\mathbb{A}^d)} \to 0.$$

The lemma follows. □

We define the *Jouanolou model* (of the structure sheaf) over $\mathrm{Conf}_I(\mathbb{A}^d)$ to be

$$\mathbf{J}^I_{\mathbb{A}^d} := \Gamma(\mathbb{J}^I_{\mathbb{A}^d}, \Omega^\bullet_{\mathbb{J}_{\mathbb{A}^d}/\mathrm{Conf}_I(\mathbb{A}^d)}). \tag{1.1}$$

From the above lemma, we have the following explicit description of the Jouanolou model

$$\mathbf{J}^I_{\mathbb{A}^d} = \frac{\mathbf{k}[z_i^s, x_{jl}^s, \mathbf{d}x_{jl}^s]^{s=1,\ldots,d}_{1\le i\le k, 1\le j<l\le k}}{\langle \sum_{s=1}^d x_{jl}^s(z_j^s - z_l^s) - 1,\ \sum_{s=1}^d \mathbf{d}x_{jl}^s(z_j^s - z_l^s)\rangle^{s=1,\ldots,d}_{1\le j<l\le k}} .$$

Here $\mathbf{d}$ is the relative differential which satisfies $\mathbf{d}(x_{jl}^s) = \mathbf{d}x_{jl}^s, \mathbf{d}z_i^s = 0$. We also use the variable x_{jl}^s for $j > l$ and declare $x_{jl}^s = -x_{lj}^s$.

Given $I \in \mathbf{fSet}$, let us denote by $\mathcal{D}_{(\mathbb{A}^d)^I}$ the sheaf of differential operators on $(\mathbb{A}^d)^I$. The space of global sections of $\mathcal{D}_{(\mathbb{A}^d)^I}$ is generated by z_i^s and $\partial_{z_i^s}$ subject to the usual relations $[\partial_{z_i^s}, z_i^s] = 1$. We will mainly deal with quasi-coherent sheaves ($\mathcal{D}$-modules) on $(\mathbb{A}^n)^k$ and to simplify notation we will not distinguish the global section $\Gamma\left((\mathbb{A}^n)^k, \mathcal{F}\right)$ and $\mathcal{F}$ itself.

Proposition 1.2. *The Jouanolou model $\mathbf{J}^I_{\mathbb{A}^d}$ has a left $\mathcal{D}_{(\mathbb{A}^d)^I}$-module structure where z_i^s acts by multiplication and*

$$[\partial_{z_j^t}, \mathbf{d}] = 0, \quad \partial_{z_j^t} x_{jl}^s = -x_{jl}^t \cdot x_{jl}^s,$$

for $1 \le s, t \le d$ and $1 \le i \le k, 1 \le j < l \le k$.

Proof. We only need to show that $\partial_{z_j^t}$ preserves the ideal $\langle \sum_{s=1}^d x_{jl}^s(z_j^s - z_l^s) - 1\rangle^{s=1,\ldots,d}_{1\le j<l\le k}$. It follows from the following direct computation

$$\begin{aligned}
&\partial_{z_j^t}\left(\sum_{s=1}^d x_{jl}^s(z_j^s - z_l^s) - 1\right) \\
&= -x_{jl}^t \cdot \sum_{s=1}^d x_{jl}^s(z_j^s - z_l^s) + x_{jl}^t \\
&= -x_{jl}^t + x_{jl}^t = 0.
\end{aligned}$$

□

Remark 1.3. The intuition behind this definition is that one can use the following coordinate transformation

$$x_{kl}^s = \frac{u_{kl}^s}{\sum\limits_{t=1}^{d} u_{kl}^t (z_k^t - z_l^t)}.$$

Definition 1.4. Given a finite set I and an $\mathcal{O}_{(\mathbb{A}^d)^I}$-module M_I, the *Jouanolou model of* M_I is the following dg $\mathcal{O}_{(\mathbb{A}^d)^I}$-module

$$\mathbf{J}_{\mathbb{A}^d}^I(M_I) := \Gamma\left(\mathbb{J}_{\mathbb{A}^d}^I, \Omega^\bullet_{\mathbb{J}_{\mathbb{A}^d}^I/\mathrm{Conf}_I(\mathbb{A}^d)} \otimes p_I^* j_I^* M_I\right) = M_I \otimes_{\mathcal{O}_{(\mathbb{A}^d)^I}} \mathbf{J}_{\mathbb{A}^d}^I.$$

Note that if M_I is a right (respectively, left) $\mathcal{D}_{(\mathbb{A}^d)^I}$-module, then $\mathbf{J}_{\mathbb{A}^d}^I(M_I)$ is a right (respectively, left) $\mathcal{D}_{(\mathbb{A}^d)^I}$-module.

Remark 1.5. One can show that $\mathbf{J}_{\mathbb{A}^d}^I(M_I)$ represents $\mathbf{R} j_{I*} j_I^* M_I$ in the derived category of $\mathcal{D}$-modules. In fact, we have the universal Jouanolou model [2, 4.1.3. (ii)] which is defined as the jet algebra $\mathcal{J}_{\mathbb{A}^d}^I := \mathscr{J}\mathbf{J}_{\mathbb{A}^d}^I = \Omega^\bullet_{\mathscr{J}\mathbf{J}_{\mathbb{A}^d}^{I,0}/\mathcal{O}_{(\mathbb{A}^d)^I}}$. In terms of the universal Jouanolou model, we obtain our Jouanolou model by imposing the relations discussed above

$$\mathbf{J}_{\mathbb{A}^d}^I = \frac{\mathcal{J}_{\mathbb{A}^d}^I}{\langle \partial_{z_i^t} x_{ij}^s = -x_{ij}^t x_{ij}^s \rangle}$$

The universal Jouanolou model is also closely related to another model for the configuration space

$$\mathbf{P}_{\mathbb{A}^d}^I = \frac{\mathcal{J}_{\mathbb{A}^d}^I}{\langle (z_i^s - z_j^s) \cdot (\partial_{z_i^s} x_{ij}^s) = x_{ij}^s \rangle}$$

called the *polysimplicial model* which is considered in [6].

The quotient maps

$$\mathbf{J}_{\mathbb{A}^d}^I \xleftarrow{\sim} \mathcal{J}_{\mathbb{A}^d}^I \xrightarrow{\sim} \mathbf{P}_{\mathbb{A}^d}^I$$

are quasi-isomorphism as $\mathcal{D}$-modules. Furthermore, one can show that they are flat $\mathcal{O}_{(\mathbb{A}^d)^I}$-algebras. This implies that

$$\mathbf{J}_{\mathbb{A}^d}^I[M] \xleftarrow{\sim} \mathcal{J}_{\mathbb{A}^d}^I[M] \xrightarrow{\sim} \mathbf{P}_{\mathbb{A}^d}^I[M].$$

The claim that $\mathbf{J}_{\mathbb{A}^d}^I[M] \sim \mathbf{R} j_{I*} j_I^* M_I$ follows from the fact that $\mathbf{P}_{\mathbb{A}^d}^I[M] \sim \mathbf{R} j_{I*} j_I^* M_I$, see [6].

Our main motivation for the Jouanolou model is in order to formulate the concept of *chiral operations* in higher dimensions. In complex dimension one, chiral operations form the operadic structure controlling *chiral algebras*. Indeed, in [2], the authors define the space of chiral operations as

$$\mathrm{Hom}_{\mathcal{D}_{X^I}}\left(j_* j^* M^{\boxtimes I}, \Delta_*^{I/\{\star\}} M\right), \quad \Delta^{I/\{\star\}} \colon (\mathbb{A}^d)^{\{\star\}} = \mathbb{A}^d \hookrightarrow (\mathbb{A}^d)^I.$$

Chiral operations form an operad for any $\mathcal{D}_{\mathbb{A}^1}$-module $M = \mathcal{A}$, which we'll denote $\mathcal{P}^{ch}_{\mathbb{A}^1}[\mathcal{A}]$. A *chiral algebra* structure on $\mathcal{A}$, following to Beilinson and Drinfeld, is a map of operads

$$\mathcal{L}\mathrm{ie} \to \mathcal{P}^{ch}_{\mathbb{A}^1}[\mathcal{A}]. \tag{1.2}$$

We conclude that the Jouanolou model $\mathbf{J}^I_{\mathbb{A}^d}(M_I)$ can be thought of as the domain of the higher-dimensional chiral operation. We now move on to spelling out the definition of a higher-dimensional chiral algebra.

1.2. **The definition of higher chiral algebras.**

Let $I \in \mathbf{fSet}$ and M be a right $\mathcal{D}_{\mathbb{A}^d}$-module. Then the space of global sections of the diagonal $\mathcal{D}$-module pushforward of M has the following expression

$$\Gamma\left((\mathbb{A}^d)^I, \Delta_*^{I/\{\star\}} M\right) = M \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I].$$

where $\Delta^{I/\{\star\}} \colon (\mathbb{A}^d)^{\{\star\}} = \mathbb{A}^d \hookrightarrow (\mathbb{A}^d)^I$ is the diagonal embedding and

$$\mathbf{k}[\lambda_\star] := \mathbf{k}[\lambda^1_\star, \dots, \lambda^d_\star], \quad \mathbf{k}[\lambda_I] := \otimes_{i\in I} \mathbf{k}[\lambda^1_i, \dots, \lambda^d_i],$$

and the tensor notation $- \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I]$ indicates the following:

- for $s = 1, \dots, d$, $\lambda^s_\star = \sum\limits_{i\in I} \lambda^s_i$;
- for $s = 1, \dots, d$, $\lambda^s_\star$ acts on M on the right as ∂_{z^s}.

The right $\mathcal{D}_{(\mathbb{A}^d)^I}$-module structure on $M \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I]$ is given by the following explicit formulas

$$n \otimes F(\lambda^s_i) \cdot z^t_k = n \cdot z^t \otimes F(\lambda^s_i) + n \otimes \partial_{\lambda^t_k} F(\lambda^s_i), \quad n \otimes F(\lambda^s_i) \cdot \partial_{z^t_k} = n \otimes F(\lambda^s_i) \lambda^t_k.$$

For a non-empty finite set I with $|I| = k$, we define the space of d-dimensional chiral I-operations, in the Jouanolou model, to be the dg vector space

$$\mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[M](I \to \{\star\}) = \mathrm{Hom}_{\mathcal{D}_{(\mathbb{A}^d)^I}}\left(\mathbf{J}^I_{\mathbb{A}^d}(M^{\boxtimes I}), \Delta_*^{I/\{\star\}} M\right).$$

These vector spaces are naturally cochain complexes in representations for the symmetric group S_k.

Definition 1.6. We define the action of the symmetric group $S_{|I|}$ on $\mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[M](I \to \{\star\})$ by

$$(\sigma\mu)(-) := \sigma\left(\mu(\sigma^{-1}-)\right)$$

where

1. the action on $\mathbf{J}^I_{\mathbb{A}^d}(M^{\boxtimes I})$ is given by
$$\sigma\left(\alpha(z^s_i, x^s_{ij}, \mathbf{d}x^s_{ij})\cdot m_1\boxtimes\cdots\boxtimes m_k\right)$$
$$:= (-1)^{\chi(m_1,\dots,m_k;\sigma)}\cdot\alpha(z^s_{\sigma(i)}, x^s_{\sigma(i)\sigma(j)}, \mathbf{d}x^s_{\sigma(i)\sigma(j)})\cdot m_{\sigma^{-1}(1)}\boxtimes\cdots\boxtimes m_{\sigma^{-1}(k)}.$$
where $m_i\in M$ and $\chi(m_1,\dots,m_k;\sigma)$ is the Koszul sign.
2. the action on $\Delta^{\{1,\dots,k\}/\{\star\}}_{*}M$ is given by
$$\sigma\left(m\otimes f(\lambda^s_1,\dots,\lambda^s_k)\right) := m\otimes f(\lambda^s_{\sigma(1)},\dots,\lambda^s_{\sigma(k)}).$$

Theorem 1.7. *For a right $\mathcal{D}_{\mathbb{A}^d}$-module M, the collection*
$$\{\mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[M](k)\}_{k\geq 0},$$
where we define the dg S_k-module
$$\mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[M](k) = \begin{cases} 0, & \text{if } k=0;\\ \mathrm{Hom}_{\mathcal{D}_{(\mathbb{A}^d)^k}}\left(\mathbf{J}^k_{\mathbb{A}^d}(M^{\boxtimes k}), \Delta^{\{1,\dots,k\}/\{\star\}}_{*}M\right), & k\geq 1.\end{cases}$$
has the structure of a dg operad.

Proof. The construction of the operad structure is completely parallel to [6]. Here we only spell out how to compose two chiral operations.

Suppose that $I = I'\bigsqcup I''$ and we are given two chiral operations $\mu_{I'}\in\mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[M](I'\twoheadrightarrow\{\bullet\})$, $\mu_{\{\bullet\}\sqcup I''}\in\mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[M](\{\bullet\}\sqcup I''\twoheadrightarrow\{\star\})$, we would like to define the corresponding partial composition
$$\mu_{\{\bullet\}\sqcup I''}\circ\mu_{I'}\in\mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[M](I\twoheadrightarrow\{\star\})$$

We first construct the following operation (by abuse of notation, we continue to use $\mu_{I'}$)
$$\mu_{I'}\in\mathrm{Hom}_{\mathcal{D}_{(\mathbb{A}^d)^I}}\left(\mathbf{J}^I_{\mathbb{A}^d}(M^{\boxtimes I}), \mathbf{J}^{\{\bullet\}\sqcup I''}_{\mathbb{A}^d}(M\boxtimes M^{\boxtimes I''})\otimes_{\mathbf{k}[\lambda_\bullet]}\mathbf{k}[\lambda_{I'}]\right).$$
We introduce a partial Jouanolou torsor
$$\mathbb{J}^{(I'),I''}_{\mathbb{A}^d} := \{\sum_{s=1}^{d} x^s_{ij}(z^s_i - z^s_j) = 1 | i,j\in I'' \text{ or } i\in I', j\in I''\}\subset(\mathbb{A}^d)^I\times(\mathbb{A}^d)^{\binom{|I''|}{2}+|I'|\cdot|I''|}.$$
The corresponding Jouanolou model $\mathbf{J}^{(I'),I''}_{\mathbb{A}^d} = \Gamma(\mathbb{J}^{(I'),I''}_{\mathbb{A}^d}, \Omega^\bullet_{\mathbb{J}^{(I'),I''}_{\mathbb{A}^d}/\mathrm{Conf}_I(\mathbb{A}^d)})$ has the following property
$$\mathbf{J}^I_{\mathbb{A}^d} = \mathbf{J}^{I'}_{\mathbb{A}^d}\otimes_{\mathcal{O}_{(\mathbb{A}^d)^{I'}}}\mathbf{J}^{(I'),I''}_{\mathbb{A}^d}.$$

First, the chiral operation $\mu_{I'}$ gives rise to map

$$\mathbf{J}^{I}_{\mathbb{A}^d}(M^{\boxtimes I}) = \mathbf{J}^{(I'),I''}_{\mathbb{A}^d}(\mathbf{J}^{I'}_{\mathbb{A}^d}(M^{\boxtimes I'}) \boxtimes M^{\boxtimes I''}) \to \mathbf{J}^{(I'),I''}_{\mathbb{A}^d}(\Delta^{I'}_{*} M \boxtimes M^{\boxtimes I''}).$$

Consider the following $\mathcal{O}_{(\mathbb{A}^d)^I}$-module

$$\left(M \boxtimes M^{\boxtimes I''}\right) \otimes_{\mathcal{O}_{(\mathbb{A}^d)^I}} \mathbf{J}^{(I'),I''}_{\mathbb{A}^d}$$

The identity map can be extended to a $\mathcal{D}$-module map

$$\begin{aligned} v^{(I')} : \mathbf{J}^{(I'),I''}_{\mathbb{A}^d}(\Delta^{I'}_{*} M \boxtimes M^{\boxtimes I''}) &= \left((M \otimes_{\mathbf{k}[\lambda_\bullet]} \mathbf{k}[\lambda_{I'}]) \boxtimes M^{\boxtimes I''}\right) \otimes_{\mathcal{O}_{(\mathbb{A}^d)^I}} \mathbf{J}^{(I'),I''}_{\mathbb{A}^d} \\ &\to \left(\left(M \boxtimes M^{\boxtimes I''}\right) \otimes_{\mathcal{O}_{(\mathbb{A}^d)^I}} \mathbf{J}^{(I'),I''}_{\mathbb{A}^d}\right) \otimes_{\mathbf{k}[\lambda_\bullet]} \mathbf{k}[\lambda_{I'}] \end{aligned}$$

We then define the map

$$c^{(I')} : \mathbf{J}^{(I'),I''}_{\mathbb{A}^d} \to \mathbf{J}^{\{\bullet\} \sqcup I''}_{\mathbb{A}^d}$$

$$\begin{cases} x^s_{ij} \mapsto x^s_{\bullet j} \\ z^s_i \mapsto z^s_\bullet \end{cases}, \quad \begin{cases} x^s_{jj'} \mapsto x^s_{jj'} \\ z^s_j \mapsto z^s_j \end{cases}, \quad s = 1, \dots, d,\ i \in I',\ j, j' \in I''$$

which is a map of $k[\lambda_\bullet]$-modules. It induces a map (by abuse of notation again)

$$c^{(I')} : \left(\left(M \boxtimes M^{\boxtimes I''}\right) \otimes_{\mathcal{O}_{(\mathbb{A}^d)^I}} \mathbf{J}^{(I'),I''}_{\mathbb{A}^d}\right) \otimes_{\mathbf{k}[\lambda_\bullet]} \mathbf{k}[\lambda_{I'}] \to \left(\left(M \boxtimes M^{\boxtimes I''}\right) \otimes_{\mathcal{O}_{(\mathbb{A}^d)^{\{\bullet\}\sqcup I''}}} \mathbf{J}^{\{\bullet\}\sqcup I''}_{\mathbb{A}^d}\right) \otimes_{\mathbf{k}[\lambda_\bullet]} \mathbf{k}[\lambda_{I'}].$$

This morphism $c^{(I')}$ is an explicit model for restriction along the diagonal.

Using the morphism $c^{(I')}$, we can construct the $\mathcal{D}$-module map

$$c^{(I')} \circ v^{(I')} : \mathbf{J}^{(I'),I''}_{\mathbb{A}^d}(\Delta^{I'}_{*} M \boxtimes M^{\boxtimes I''}) \to \mathbf{J}^{\{\bullet\}\sqcup I''}_{\mathbb{A}^d}(M \boxtimes M^{\boxtimes I''}) \otimes_{\mathbf{k}[\lambda_\bullet]} \mathbf{k}[\lambda_{I'}]$$

To summarize, we have

$$\mu_{I' \subset I} : \mathbf{J}^{I}_{\mathbb{A}^d}(M^{\boxtimes I}) = \mathbf{J}^{I' \bigsqcup I''}_{\mathbb{A}^d}(M^{\boxtimes I' \bigsqcup I''}) \xrightarrow{\mu_{I'} \boxtimes \mathrm{Id}} \mathbf{J}^{(I'),I''}_{\mathbb{A}^d}(\Delta^{I'}_{*} M \boxtimes M^{\boxtimes I''}) \xrightarrow{c^{(I')} \circ v^{(I')}} \mathbf{J}^{\{\bullet\}\sqcup I''}_{\mathbb{A}^d}(M \boxtimes M^{\boxtimes I''}) \otimes_{\mathbf{k}[\lambda_\bullet]} \mathbf{k}[\lambda_{I'}].$$

We can further compose $\mu_{I' \subset I}$ with $\mu_{\bullet \bigsqcup \vec{I''}} \in \mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[M](\{\bullet\} \bigsqcup \vec{I''} \twoheadrightarrow \{\star\})$:

$$\begin{aligned} \mu_{\bullet \bigsqcup I''} \circ \mu_{I' \subset I} : \mathbf{J}^{I}_{\mathbb{A}^d}(M^{\boxtimes I}) \xrightarrow{\mu_{I' \subset I}} \mathbf{J}^{\{\bullet\}\sqcup I''}_{\mathbb{A}^d}(M \boxtimes M^{\boxtimes I''}) \otimes_{\mathbf{k}[\lambda_\bullet]} \mathbf{k}[\lambda_{I'}] &\xrightarrow{\mu_{\bullet \sqcup I''}} M \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_{\{\bullet\}\cup I''}] \otimes_{\mathbf{k}[\lambda_\bullet]} \mathbf{k}[\lambda_{I'}] \\ &= M \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I]. \end{aligned}$$

By slightly abuse of notation we define $\mu_{\{\bullet\}\sqcup I''} \circ \mu_{I'} := \mu_{\bullet \bigsqcup I''} \circ \mu_{I' \subset I}$. □

We conclude this section with our main definition.

Definition 1.8. A *homotopy chiral algebra* in the Jouanolou model is a dg $\mathcal{D}_{\mathbb{A}^d}$-module $\mathcal{A}$ together with a map of dg operads

$$\mathcal{L}\mathrm{ie}_\infty \to \mathcal{P}^{ch,\mathbf{J}}_{\mathbb{A}^d}[\mathcal{A}]$$

where $\mathcal{L}\mathrm{ie}_\infty$ is the dg operad controlling L_∞ algebras.

Explicitly, then, to construct a chiral algebra we must product a collection of $\mathcal{D}$-module maps $\{\mu_I\}$ which satisfy the L_∞ relations.

Remark 1.9. A homotopy chiral algebra in the Jouanolou defines a d-dimensional chiral algebra, up to equivalence, in the sense of [7].

Remark 1.10. In dimension $d = 1$, the notion of an L_∞ chiral algebra has recently appeared in [18]. When $d = 1$, our definition reduces to that of *loc. cit.*.

Remark 1.11. Using the method of Thom-Sullivan-Whitney for computing homotopy limits of cosimplicial spaces a different model for configuration spaces has recently been presented in [6]. The notion of a chiral algebra in [6] is equivalent, in a homotopical sense, is equivalent to the notion we present here. On the other hand, explicit examples are presented quite differently in each respective model. For instance, it is clear from our constructions that the examples of chiral algebras which appear in section 3 depend continuously on the input data. Additionally, the Jouanolou model has a GL_d-symmetry which endows the Jouanolou operad with a GL_d-equivariant structure. Furthermore, our later construction of the unit chiral algebra is GL_d-equivariant, which is not true for the model given in [6].

Another advantage of our approach is that chiral operations in the Jouanolou model are manifestly related to holomorphic quantum field theory; indeed, our construction of the *unit* chiral algebra uses Feynman graph integrals as we will soon explain. Going the opposite direction, we will see in section 4 how our chiral operations can be used to recover explicit formulas for expectation values in higher-dimensional quantum field theory.

Before we move on, we point out a general construction which mimics a construction for ordinary, $d = 1$, chiral algebras. Let $h(-)$ denote the de Rham cohomology functor. Thus, to a dg $\mathcal{D}$-module $\mathcal{A}$, we obtain the sheaf $h(\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{D}} \mathcal{O}$. For $\mathcal{A}$ a chiral algebra in the Jouanolou model the chiral operations endow $h(\mathcal{A})$ with the structure of a sheaf of L_∞ algebras.

Let $I = \{1, \ldots, m\}$. Observe that there is a natural map

$$\mathcal{A}^{\boxtimes m} \to \mathbf{J}^I_{\mathbb{A}^d}(\mathcal{A}^{\boxtimes I}). \tag{1.3}$$

We can post compose this map with the chiral operation μ_I to obtain the map of dg $\mathcal{D}$-modules

$$\mathcal{A}^{\boxtimes m} \to \Delta_*^{I/\{\star\}} \mathcal{A} \tag{1.4}$$

Applying the de Rham functor $h(-)$, we obtain the m-linear map

$$h(\mathcal{A}\boxtimes\cdots\boxtimes\mathcal{A})\to\Delta^{I}_{\cdot}h(\mathcal{A}) \tag{1.5}$$

where now $\Delta_{\cdot}$ denotes the sheaf-theoretic pushforward. Applying adjunction, we obtain the m-linear map

$$[-]_m\colon h(\mathcal{A})\times\cdots\times h(\mathcal{A})\to h(\mathcal{A}).$$

The L_∞ relations which are satisfied by the chiral operations $\{\mu_I\}$ defining $\mathcal{A}$ immediately yield the following.

Proposition 1.12. *The operations $\{[-]_m\}$ endow $h(\mathcal{A})$ with the structure of an L_∞ algebra.*

2. FEYNMAN GRAPH INTEGRAL CONSTRUCTION

In this section, we will use Feynman graph integrals in the setting of holomorphic quantum field theory to construct the unit L_∞ chiral operations in any dimension. In section 2.1, we recall the standard chiral operation in dimension $d=1$. Of course, here, the chiral operations of the unit chiral algebra are strict in the L_∞ sense. In higher dimensions, we largely use the techniques developed in [19].

We will work over $\mathbf{k}=\mathbb{C}$ throughout this section. The affine variety $\mathbb{A}^d$ is understood as $\mathbb{C}^d$ with analytic topology. For example, $\mathcal{O}_{\mathbb{A}^d}$ is the sheaf of holomorphic functions on $\mathbb{C}$.

Theorem 2.1. *The $\mathcal{D}$-module $\omega_{\mathbb{A}^d}[d-1]$ carries the structure of a GL_d-equivariant homotopy chiral algebra in the Jouanolou model. For each k, its k-ary operation is expressed in terms of the higher dimensional residue, see definition 2.19. It is a model for the unit homotopy chiral algebra.*

2.1. The unit chiral algebra on $\mathbb{A}^1$.

We begin by reviewing the one-dimensional unit chiral algebra $\omega_{\mathbb{A}^1}$ and how Feynman graph integrals can be used to repackage the chiral operations. Our most novel viewpoint is the utility of Schwinger space integrals (see appendix B) to express these operations. This approach will apply more generally and allow us to construct the unit chiral operations in higher dimensions.

When $d=1$, the model for the source of two-ary chiral operations for $\omega_{\mathbb{A}^1}$ is

$$\mathbf{J}^{\{1,2\}}_{\mathbb{A}^1}(\omega_{\mathbb{A}^1}^{\boxtimes\{1,2\}})=\left\{f(z_1,z_2)dz_1\boxtimes dz_2\mid f\in k\left[z_1,z_2,\frac{1}{z_1-z_2}\right]\right\}$$

while the target for the two-ary chiral operations is

$$\omega_{\mathbb{A}^1}\otimes_{k[\lambda_\star]}k[\lambda_1,\lambda_2] \tag{2.1}$$

where $\lambda_\star$ acts on $\omega_{\mathbb{A}^1}$ by ∂_z and, on the left, acts by $\lambda_\star=\lambda_1+\lambda_2$.

Definition 2.2. We define the chiral operation $\mu_2 = \mu_{\{1,2\}}$ for the unit $\omega_{\mathbb{A}^1}$ by the following formula:

$$\mu_2(f(z_1,z_2)dz_1 \boxtimes dz_2) = \frac{1}{2\pi i} e^{-(\lambda_1+\lambda_2)w} \left(\oint_{z_1=z_2} f(z_1,z_2) e^{\lambda_1 z_1 + \lambda_2 z_2} dz_1 \boxtimes dz_2 \right)\Big|_{z_2=w},$$

where $\oint_{z_1=z_2}$ is the contour integral of variable z_1 over a simple loop around z_2.

We unpack the definition. Suppose that

$$\alpha = \frac{g(z_1,z_2)}{(z_1-z_2)^n} dz_1 \boxtimes dz_2 \in \mathbf{J}^{\{1,2\}}_{\mathbb{A}^1}(\omega_{\mathbb{A}^1}^{\boxtimes\{1,2\}}),$$

where $g(z_1,z_2) \in k[z_1,z_2]$. Expanding, we find that

$$\begin{aligned}
\mu_2(\alpha) &= \frac{1}{2\pi i} e^{-(\lambda_1+\lambda_2)w} \left(\oint_{z_1=z_2} \frac{g(z_1,z_2)}{(z_1-z_2)^n} e^{\lambda_1 z_1 + \lambda_2 z_2} dz_1 \boxtimes dz_2 \right)\Big|_{z_2=w} \\
&= e^{-(\lambda_1+\lambda_2)w} \frac{1}{(n-1)!} \left(\partial_{z_1}^{n-1} \left(g(z_1,w) e^{\lambda_1 z_1 + \lambda_2 w} \right) \right)\Big|_{z_1=w} dw \\
&= \frac{1}{(n-1)!} \left((\partial_{z_1} + \lambda_1)^{n-1} g(z_1,w) \right)\Big|_{z_1=w} dw,
\end{aligned}$$

so $\mu_2(\alpha) \in \omega_{\mathbb{A}^1} \otimes_{k[\lambda_\star]} k[\lambda_1,\lambda_2]$. For example, when $g = 1$ this reduces to

$$\mu_2\left(\frac{dz_1 \boxtimes dz_2}{(z_1-z_2)^n} \right) = \frac{\lambda_1^{n-1}}{(n-1)!} \otimes dw \tag{2.2}$$

Proposition 2.3. *μ_2 defines a chiral operation, i.e.,*

$$\mu_2 \in \mathcal{P}^{ch}_{\mathbb{A}^1}[\omega_{\mathbb{A}^1}](2)$$

Proof. We prove that μ_2 is a $\mathcal{D}_{(\mathbb{A}^1)^2}[(\mathbb{A}^1)^2]$-module morphism.

Recall that the D-module structure is determined by the rules

$$\alpha \cdot z_i = z_i \alpha, \quad \alpha \cdot \partial_{z_i} = -\partial_{z_i} \left(\frac{g(z_1,z_2)}{(z_1-z_2)^n} \right) dz_1 \boxtimes dz_2.$$

Thus, we can directly compute:

$$\begin{aligned}
\mu_2(\alpha \cdot z_i) &= \frac{1}{2\pi i} e^{-(\lambda_1+\lambda_2)w} \left(\oint_{z_1=z_2} \frac{z_i g(z_1,z_2)}{(z_1-z_2)^n} e^{\lambda_1 z_1 + \lambda_2 z_2} dz_1 \boxtimes dz_2 \right)\Big|_{z_2=w} \\
&= \frac{1}{2\pi i} e^{-(\lambda_1+\lambda_2)w} \left(\oint_{z_1=z_2} \frac{g(z_1,z_2)}{(z_1-z_2)^n} \partial_{\lambda_i} \left(e^{\lambda_1 z_1 + \lambda_2 z_2} \right) dz_1 \boxtimes dz_2 \right)\Big|_{z_2=w} \\
&= \frac{1}{2\pi i} (w + \partial_{\lambda_i}) \left(e^{-(\lambda_1+\lambda_2)w} \left(\oint_{z_1=z_2} \frac{g(z_1,z_2)}{(z_1-z_2)^n} e^{\lambda_1 z_1 + \lambda_2 z_2} dz_1 \boxtimes dz_2 \right)\Big|_{z_2=w} \right) \\
&= \mu_2(\alpha) \cdot z_i,
\end{aligned}$$

$$\begin{aligned}
\mu_2(\alpha\cdot\partial_{z_i}) &= -\frac{1}{2\pi i}e^{-(\lambda_1+\lambda_2)w}\left(\oint_{z_1=z_2}\partial_{z_i}\left(\frac{g(z_1,z_2)}{(z_1-z_2)^n}\right)e^{\lambda_1z_1+\lambda_2z_2}dz_1\boxtimes dz_2\right)\Big|_{z_2=w}\\
&= -\frac{1}{2\pi i}e^{-(\lambda_1+\lambda_2)w}\left(\oint_{z_1=z_2}\partial_{z_i}\left(\frac{g(z_1,z_2)}{(z_1-z_2)^n}e^{\lambda_1z_1+\lambda_2z_2}\right)dz_1\boxtimes dz_2\right)\Big|_{z_2=w}\\
&+\frac{1}{2\pi i}e^{-(\lambda_1+\lambda_2)w}\left(\oint_{z_1=z_2}\frac{g(z_1,z_2)}{(z_1-z_2)^n}\partial_{z_i}\left(e^{\lambda_1z_1+\lambda_2z_2}\right)dz_1\boxtimes dz_2\right)\Big|_{z_2=w}\\
&= 0+\frac{\lambda_i}{2\pi i}e^{-(\lambda_1+\lambda_2)w}\left(\oint_{z_1=z_2}\frac{g(z_1,z_2)}{(z_1-z_2)^n}e^{\lambda_1z_1+\lambda_2z_2}dz_1\boxtimes dz_2\right)\Big|_{z_2=w}\\
&= \mu_2(\alpha)\cdot\partial_{z_i}.
\end{aligned}$$

□

The special feature in dimension one is that the unit chiral algebra has trivial higher operations. Notice that by simple degree reasons, the unit chiral algebra must necessarily have vanishing $k\geq 3$-ary operations. In fact, the operation μ_2 satisfies the ordinary Jacobi identity.

Proposition 2.4 ([2], see also [13])**.** *The operation μ_2 satisfies the Jacobi identity, hence it defines the structure of a (strict, one-dimensional) chiral algebra.*

In the definition of chiral operation for $\omega_{\mathbb{A}^1}$, we used the residue. Now we introduce an alternative method to define the residue. The advantage of this definition is that it lends itself to a natural higher-dimensional extension. Thus, this is the approach we will take in the next section to define the higher-dimensoinal chiral operations (for the unit chiral algebra). The first step is to prove the following integral representation of the Cauchy kernel.

Lemma 2.5.

$$\frac{1}{z}=-\int_0^{+\infty}e^{-zy}dy,$$

where $y=\frac{\bar z}{t}$, and dy is viewed as the differential form

$$dy=\frac{1}{t}d\bar z-\frac{\bar z}{t^2}dt.$$

The integration is over t. We call t the Schwinger parameter.

Proof.

$$\frac{1}{z}=\frac{\bar z}{z\bar z}=\int_0^{+\infty}\bar z e^{-z\bar z t}dt=-\int_0^{+\infty}e^{-\frac{z\bar z}{t}}\bar z d\left(\frac{1}{t}\right)=-\int_0^{+\infty}e^{-zy}dy. \tag{2.3}$$

□

Let $\alpha = \frac{g(z)}{z^n}dz$, we can use above lemma to rephrase the residue:

$$\begin{aligned}\oint_{z=0} \alpha = \oint_{z=0} \frac{g(z)}{z^n} dz \\ = \int_{(0,+\infty)^n} \int_{S^1} e^{-\left(\Sigma_{k=1}^{n} \frac{1}{t_k}\right) z\bar{z}} g(z) \prod_{k=1}^{n} dy_k dz,\end{aligned}$$

where $S^1 = \{z \in k = \mathbb{C} | |z|^2 = 1\}$, $y_k = \frac{\bar{z}}{t_k}$.

Let $d_{\mathbb{A}^1}$, d_t be the de Rham differential on $\mathbb{A}^1$ and $(0, +\infty)^n$ respectively. Then, the differential form appearing in the above formula is $(d_{\mathbb{A}^1} + d_t)$-closed:

$$(d_{\mathbb{A}^1} + d_t) \left(e^{-\left(\Sigma_{k=1}^{n} \frac{1}{t_k}\right) z\bar{z}} g(z) \prod_{k=1}^{n} dy_k dz \right) = 0.$$

We will find an equivalent way to express the residue $\oint_{z=0} \alpha$ in terms of a homologous integration cycle.

Let

$$S^+((0, +\infty)^n) = \{(t_1, \cdots, t_n) \in (0, +\infty)^n | \sum_{k=1}^{n} t_k^2 = 1\} \subset (0, +\infty)^n,$$

by using Stokes theorem carefully, we can get

$$\begin{aligned}\oint_{z=0} \alpha = \int_{(0,+\infty)^n} \int_{S^1} e^{-\left(\Sigma_{k=1}^{n} \frac{1}{t_k}\right) z\bar{z}} g(z) \prod_{k=1}^{n} dy_k dz \\ = \int_{S^+((0,+\infty)^n)} \int_{\mathbb{A}^1} e^{-\left(\Sigma_{k=1}^{n} \frac{1}{t_k}\right) z\bar{z}} g(z) \prod_{k=1}^{n} dy_k dz.\end{aligned}$$

Rather than constructing a homotopy between this two integration cycles, we provide a direct computation to obtain this result:

Proposition 2.6. *Let $\alpha = \frac{g(z)}{z^n}dz$, we have*

$$\frac{1}{2\pi i} \int_{S^+((0,+\infty)^n)} \int_{\mathbb{A}^1} e^{-\left(\Sigma_{k=1}^{n} \frac{1}{t_k}\right) z\bar{z}} g(z) \prod_{k=1}^{n} dy_k dz = \frac{\partial_z^{n-1} g(0)}{(n-1)!}.$$

Proof. Noticing the integral of Gaussian type, we can apply Wick's theorem to obtain (2.4)

$$\frac{1}{2\pi i} \int_{\mathbb{A}^1} e^{-\left(\Sigma_{k=1}^{n} \frac{1}{t_k}\right) z\bar{z}} g(z) \prod_{k=1}^{n} dy_k dz = (-1)^{n-1} \frac{1}{\Sigma_{k=1}^{n} \frac{1}{t_k}} \iota_{\partial_{\bar{z}}} e^{\frac{1}{\Sigma_{k=1}^{n} \frac{1}{t_k}} \partial_z \partial_{\bar{z}}} \left(g(z) \prod_{k=1}^{n} dy_k \right) \Bigg|_{z=0},$$

where $\iota_{\partial_{\bar{z}}}$ denotes the operation of taking the interior product of the vector field $\partial_{\bar{z}}$ with a differential form.

Notice that as operators acting on differential forms, we have the following relation

$$e^{\frac{1}{\sum_{k=1}^n \frac{1}{t_k}}\partial_z\partial_{\bar{z}}} \circ (dy_i \wedge -) \circ e^{\frac{-1}{\sum_{k=1}^n \frac{1}{t_k}}\partial_z\partial_{\bar{z}}} = dy_i \wedge + d(\frac{\frac{1}{t_i}}{\sum_{k=1}^n \frac{1}{t_k}})\partial_z.$$

So, we can express the Gaussian integration as

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\mathbb{A}^1} e^{-\left(\sum_{k=1}^n \frac{1}{t_k}\right)z\bar{z}} g(z)\prod_{k=1}^n dy_k dz \\ &= (-1)^{n-1}\sum_{k=1}^n \left((-1)^{k-1}\frac{\frac{1}{t_k}}{\sum_{k''=1}^n \frac{1}{t_{k''}}}\prod_{k'\neq k} d(\frac{\frac{1}{t_{k'}}}{\sum_{k''=1}^n \frac{1}{t_{k''}}})\right)\partial_z^{n-1}g(0). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{2\pi i}\int_{S^+((0,+\infty)^n)}\int_{\mathbb{A}^1} e^{-\left(\sum_{k=1}^n \frac{1}{t_k}\right)z\bar{z}} g(z)\prod_{k=1}^n dy_k dz \\ &= (-1)^{n-1}\partial_z^{n-1}g(0)\int_{S^+((0,+\infty)^n)}\sum_{k=1}^n \left((-1)^{k-1}\frac{\frac{1}{t_k}}{\sum_{k''=1}^n \frac{1}{t_{k''}}}\prod_{k'\neq k} d(\frac{\frac{1}{t_{k'}}}{\sum_{k''=1}^n \frac{1}{t_{k''}}})\right). \end{aligned}$$

To compute the integral over $S^+((0,+\infty)^n)$, we apply localization to the following $\mathbb{R}_+ = (0,+\infty)$-action on $(0,+\infty)^n$:

$$\lambda\cdot(t_1,\cdots,t_n) = (\lambda t_1,\cdots,\lambda t_n), \quad \lambda\in\mathbb{R}_+.$$

We notice that

$$\sum_{k=1}^n \left((-1)^{k-1}\frac{\frac{1}{t_k}}{\sum_{k''=1}^n \frac{1}{t_{k''}}}\prod_{k'\neq k} d(\frac{\frac{1}{t_{k'}}}{\sum_{k''=1}^n \frac{1}{t_{k''}}})\right)$$

induces a well-defined differential form on $(0,+\infty)^n/\mathbb{R}_+$, so we have

$$\begin{aligned} &(-1)^{n-1}\int_{S^+((0,+\infty)^n)}\sum_{k=1}^n \left((-1)^{k-1}\frac{\frac{1}{t_k}}{\sum_{k''=1}^n \frac{1}{t_{k''}}}\prod_{k'\neq k} d(\frac{\frac{1}{t_{k'}}}{\sum_{k''=1}^n \frac{1}{t_{k''}}})\right) \\ &= (-1)^{n-1}\int_{\{(t_1,\cdots,t_n)\in((0,+\infty)^n)|\sum_{k=1}^n \frac{1}{t_k}=1\}}\sum_{k=1}^n \left((-1)^{k-1}\frac{\frac{1}{t_k}}{\sum_{k''=1}^n \frac{1}{t_{k''}}}\prod_{k'\neq k} d(\frac{\frac{1}{t_{k'}}}{\sum_{k''=1}^n \frac{1}{t_{k''}}})\right) \\ &= \int_{\{(u_1,\cdots,u_n)\in((0,+\infty)^n)|\sum_{k=1}^n u_k=1\}}\sum_{k=1}^n \left((-1)^{k-1}u_k\prod_{k'\neq k} du_{k'}\right) \\ &= \frac{1}{(n-1)!}. \end{aligned}$$

Thus, the conclusion follows. □

Although the expression

$$\frac{1}{2\pi i}\int_{S^+((0,+\infty)^n)}\int_{\mathbb{A}^1} e^{-\left(\sum_{k=1}^n \frac{1}{t_k}\right)z\bar{z}} g(z)\prod_{k=1}^n dy_k dz$$

appears to be more complicated than the purely holomorphic contour integral

$$\frac{1}{2\pi i}\oint_{z=0}\frac{g(z)}{z^n}dz,$$

we will show that it can be extended to higher dimensional affine spaces in the next subsection.

Remark 2.7. This computation is parallel to the calculation in [17], where the author identifies the QME in BV formalism with a Maurer-Cartan equation in a vertex algebra.

2.2. **Holomorphic Feynman graph integrals.** In this subsection, we lay out the prerequisite work on Feynman graph integrals which are central to the concept of residues that we develop for the Jouanolou model of $(\omega_{\mathbb{A}^d}[d])^{\boxtimes I}$.

Let $\mathcal{A}^{0,\bullet}(\mathrm{Conf}_I(\mathbb{A}^d))$ be the Dolbeault complex of the structure sheaf $\mathcal{O}(\mathrm{Conf}_I(\mathbb{A}^d))$. Define the map

$$i : \mathbf{J}^I_{\mathbb{A}^d} \to \mathcal{A}^{0,\bullet}(\mathrm{Conf}_I(\mathbb{A}^d))$$
$$i(z_i^s) = z_i^s, \quad i(x_{ij}^s) = \frac{\bar{z}_i^s - \bar{z}_j^s}{|z_i - z_j|^2}, \quad i(\mathbf{d}x_{ij}^s) = \bar{\partial}\left(\frac{\bar{z}_i^s - \bar{z}_j^s}{|z_i - z_j|^2}\right).$$

Here:

- $s \in \{1, 2, \ldots, d\}$ is the coordinate index.
- $i, j \in I$ are elements in the ordered set.
- $|z_i - z_j|^2 = \sum\limits_{t=1}^{d}(z_i^t - z_j^t)(\bar{z}_i^t - \bar{z}_j^t)$ is the squared norm.
- $\bar{\partial}$ is the Dolbeault differential.

It is easily checked that this map is well-defined and is a morphism of commutative dg algebras. The map i is dense in cohomology.[2]

We also point out that i is a $\mathcal{D}_{(\mathbb{A}^d)^I}$-module morphism. Similarly, we can define a natural dg $\mathcal{D}_{(\mathbb{A}^d)^I}$-module morphism

$$\mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I}) \to \mathcal{A}^{0,\bullet}(\mathrm{Conf}_I(\mathbb{A}^d), (\omega_{\mathbb{A}^d}[d])^{\boxtimes I}).$$

In the following discussion, we identify the elements of $\mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I})$ with the corresponding elements in $\mathcal{A}^{0,\bullet}(\mathrm{Conf}_I(\mathbb{A}^d), (\omega_{\mathbb{A}^d}[d])^{\boxtimes I})$.

---

[2] In fact, i is injective, but we will not need that.

2.2.1. *Schwinger space.* In our previous discussion of residues when $d = 1$, we represented the residue kernel $\frac{1}{z}$ as an integral over Schwinger parameters. Now our goal is to introduce similar integral representations opf $i(x^s_{ij})$ and $i(\mathbf{d}x^s_{ij})$:

Lemma 2.8. *We have the following equalities:*

(1)

$$i(x^s_{ij}) = -\int_0^{+\infty} e^{-(z_{ij}|y_{ij})} dy^s_{ij}$$

(2)

$$i(\mathbf{d}x^s_{ij}) = -\int_0^{+\infty} e^{-(z_{ij}|y_{ij})} (z_{ij}|dy_{ij}) dy^s_{ij}$$

Here:

- $y^s_{ij} = \frac{\bar{z}^s_i - \bar{z}^s_j}{t}$, *where* t *is called the Schwinger parameter.*
- $z^s_{ij} = z^s_i - z^s_j$.
- $(z_{ij}|y_{ij}) = \sum\limits_{s=1}^{d} z^s_{ij} y^s_{ij}$ *is the dot product.*
- dy^s_{ij} *is the differential form*

$$dy^s_{ij} = \frac{d\bar{z}^s_i - d\bar{z}^s_j}{t} - \frac{(\bar{z}^s_i - \bar{z}^s_j)dt}{t^2}.$$

- *the integrations are both with respect to the Schwinger parameter* t.

Proof. We note

$$\begin{aligned} i(x^s_{ij}) &= \frac{\bar{z}^s_i - \bar{z}^s_j}{|z_i - z_j|^2} \\ &= (\bar{z}^s_i - \bar{z}^s_j) \int_0^{+\infty} e^{-|z_i - z_j|^2 u} du \\ &= -(\bar{z}^s_i - \bar{z}^s_j) \int_0^{+\infty} e^{-\frac{|z_i - z_j|^2}{t}} d\left(\frac{1}{t}\right) \\ &= -\int_0^{+\infty} e^{-(z_{ij}|y_{ij})} dy^s_{ij}. \end{aligned}$$

In the last line we have used the fact that only the dt component contributes to the integral.

By using dominated convergence theorem for derivatives, We can interchange the order of integration and differentiation when $z_i \neq z_j$. So

$$\begin{aligned} i(\mathbf{d}x_{ij}^s) &= -\bar{\partial}\left(\int_0^{+\infty} e^{-(z_{ij}|y_{ij})} dy_{ij}^s\right) \\ &= \int_0^{+\infty} \bar{\partial}\left(e^{-(z_{ij}|y_{ij})} dy_{ij}^s\right) \\ &= \int_0^{+\infty} (\bar{\partial} + d_t)\left(e^{-(z_{ij}|y_{ij})} dy_{ij}^s\right) \\ &= -\int_0^{+\infty} e^{-(z_{ij}|y_{ij})} (z_{ij}|dy_{ij}) dy_{ij}^s, \end{aligned}$$

where d_t is the de Rham differential over the Schwinger parameter space. Again, we emphasize that only the term proportional to dt will contribute to the integral. □

The following proposition is useful and is a direct computation.

Proposition 2.9. *For $i, j \in I$ define the vector field*

$$V_{ij} = \sum_{s=1}^{d} (\bar{z}_i^s \partial_{\bar{z}_i^s} + \bar{z}_j^s \partial_{\bar{z}_j^s}) + t\partial_t.$$

If $L_{\vec{V}_{ij}}$ and $\iota_{\vec{V}_{ij}}$ denote the Lie derivative and interior product operator, respectively, then we have the following identities:

$$\begin{cases} L_{V_{ij}} e^{-(z_{ij}|y_{ij})} dy_{ij}^s &= 0, \\ L_{V_{ij}} e^{-(z_{ij}|y_{ij})} \cdot (z_{ij}|y_{ij}) dy_{ij}^s &= 0, \\ \iota_{V_{ij}} e^{-(z_{ij}|y_{ij})} dy_{ij}^s &= 0, \\ \iota_{V_{ij}} e^{-(z_{ij}|y_{ij})} \cdot (z_{ij}|y_{ij}) dy_{ij}^s &= 0. \end{cases}$$

2.2.2. *Feynman graph integrals.* Next, we introduce a graphical language to encode elements in the Jouanolou model. To this end, we introduce some concepts from graph theory.

Definition 2.10. A *directed graph* $\vec{\Gamma}$ consists of the following data:

(1) A set $\vec{\Gamma}_0$ of vertices. We use $|\Gamma_0|$ to denote the number of vertices.
(2) An ordered set $\vec{\Gamma}_0$ of directed edges. We use $|\Gamma_1|$ to denote the number of directed edges.
(3) Two maps

$$t, h : \Gamma_1 \to \Gamma_0,$$

which are assignments of tail and head to each directed edge. We require that

$$t(e) \neq h(e)$$

for any $e \in \vec{\Gamma}_1$, i.e., the graph $\vec{\Gamma}$ has no self-loops.

Furthermore, we say that a directed graph $\vec{\Gamma}$ is *decorated* if we have a special edge $e_l \in \vec{\Gamma}_1$ and a map

$$m \colon \Gamma_1 \to \{1, 2, \dots, d\}.$$

We will use $(\vec{\Gamma}, m, l)$ to denote a decorated directed graph. For simplicity, we will also call a decorated directed graph a *Feynman graph* in this article.

If $\vec{\Gamma}$ is a directed graph, we may use the following notations to describe $\vec{\Gamma}_0$ and $\vec{\Gamma}_1$:

$$\begin{cases} \vec{\Gamma}_0 = \{1 < \cdots < n = |\Gamma_0|\}, \\ \vec{\Gamma}_1 = \{e_1 < \cdots < e_{|\Gamma_1|}\}. \end{cases}$$

Definition 2.11. Let $(\vec{\Gamma}, m, l)$ be a Feynman graph, we define

$$p_{(\vec{\Gamma},m,l)}(x_e^{m(e)}, \mathbf{d}x_e^{m(e)}) = \prod_{e < e_l \in \vec{\Gamma}_1} x_e^{m(e)} \prod_{e_l \leq e' \in \vec{\Gamma}_1} \mathbf{d}x_{e'}^{m(e')} \in \mathbf{k}[x_{ij}^s, \mathbf{d}x_{ij}^s]_{i<j\in\vec{\Gamma}_0}^{1\leq s\leq d},$$

where the order of the factors is determined by the order of $\vec{\Gamma}_1$. We have used the convention

$$x_e^{m(e)} = x_{t(e)h(e)}^{m(e)} = -x_{h(e)t(e)}^{m(e)}.$$

We call $p_{(\vec{\Gamma},m,l)}$ the corresponding monomial of $(\vec{\Gamma}, m, l)$.

As we vary over all Feynman graphs $\{p_{(\vec{\Gamma},m,l)}\}$ spans $\mathbf{k}[x_{ij}^s, \mathbf{d}x_{ij}^s]_{i<j\in\vec{\Gamma}_0}^{1\leq s\leq d}$, so we can use Feynman graphs to represent elements in $\mathbf{k}[x_{ij}^s, \mathbf{d}x_{ij}^s]_{i<j\in\vec{\Gamma}_0}^{1\leq s\leq d}$.

Definition 2.12. Let $\vec{\Gamma}$ be a directed graph. The *incidence matrix* $\rho = (\rho_{ei})_{e\in\vec{\Gamma}_1, i\in\vec{\Gamma}_0}$ is

$$\rho_{ei} = \begin{cases} 1, & \text{if } t(e) = i, \\ -1, & \text{if } s(e) = i, \\ 0, & \text{otherwise.} \end{cases}$$

The *weighted Laplacian matrix* is the $\Gamma_0 \times \Gamma_0$ matrix defined by

$$M_{\vec{\Gamma}}(t)_{ij} = \sum_{e\in\vec{\Gamma}_1} \rho_{ei} \frac{1}{t_e} \rho_{ej}$$

which is defined for $t = (t_e) \in (0, +\infty)^{\Gamma_1}$.

Proposition 2.13. *Let $\vec{\Gamma}$ be a connected directed graph. Then the matrix $M_{\vec{\Gamma}}(t) = (M_{\vec{\Gamma}}(t)_{ij})_{i,j\in\vec{\Gamma}_0 - \{n\}}$ is invertible. We use $M_{\vec{\Gamma}}^{-1}(t) = (M_{\vec{\Gamma}}^{-1}(t)_{ij})_{i,j\in\vec{\Gamma}_0-\{n\}}$ to denote the inverse matrix.*

Proof. See Corollary A.7 for an explicit description of its inverse. □

Thus, we can define the following graphical Green's function $d_{\vec{\Gamma}}^{-1}(t) = (d_{\vec{\Gamma}}^{-1}(t)_{ei})_{e\in\vec{\Gamma}_1, i\in\vec{\Gamma}_0-\{n\}}$ by the formula

$$d_{\vec{\Gamma}}^{-1}(t)_{ei} = \sum_{j=1}^{n-1} \frac{1}{t_e} \rho_{ej} M_{\vec{\Gamma}}^{-1}(t)_{ji}.$$

which is defined for $t \in (0, +\infty)^{\Gamma_1}$.

Using these notations, we can finally describe the type of integrals that we are interested in.

Definition 2.14. Let $(\vec{\Gamma}, m, l)$ be a Feynman graph and denote $\Gamma_0 = I$. Given

$$\beta \in (\omega_{\mathbb{A}^d}[d])^{\boxtimes I}$$

and

$$e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)} = e^{\sum\limits_{s=1}^{d}\sum\limits_{i=1}^{n} z_i^s \lambda_i^s} \in C^\infty((\mathbb{A}^d)^I \times (\mathbb{A}^d)^I),$$

the Feynman graph integrand is defined by

$$\begin{aligned} &W((\vec{\Gamma}, m, l), \beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) \\ &= \int_{(\mathbb{A}^d)^{I-\{n\}}} p_{(\vec{\Gamma},m,l)}(-e^{-(z_e|y_e)} dy_e^{m(e)}, -e^{-(z_e|y_e)}(z_e|dy_e) dy_e^{m(e)}) e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)} \otimes \beta, \end{aligned}$$

where $z_e = z_{t(e)} - z_{h(e)}$, $y_e = \frac{\bar{z}_{t(e)} - \bar{z}_{h(e)}}{t_e}$.

The following proposition is a result of Wick's formula for Gaussian integrals.

Proposition 2.15. *For each $t = (t_e) \in (0, +\infty)^{\Gamma_1}$ the integral*

$$W((\vec{\Gamma}, m, l), \beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)})$$

over $(\mathbb{A}^d)^{I-\{n\}}$ converges. Moreover,

(1) *If $\vec{\Gamma}$ is disconnected,*

$$W((\vec{\Gamma}, m, l), \beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) = 0.$$

(2) *If $\vec{\Gamma}$ is connected, we have the following explicit formula:*

$$\begin{aligned} &W((\vec{\Gamma}, m, l), \beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) \\ &= (-2\pi i)^{d(n-1)} \Big(\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)} \circ p_{(\vec{\Gamma},m,l)}(-d\hat{y}_e^{m(e)}, -(\sum_{s=1}^{d}(\partial_{\lambda_e^s} + \tilde{z}_e^s) \circ d\hat{y}_e^s) \circ d\hat{y}_e^{m(e)}) \\ &(1 \otimes \beta))|_{\tilde{z}_i=0, 1\le i\le n-1} \, e^{(\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)}. \end{aligned}$$

In this proposition, we have introduced the following notations:

- We introduce the following coordinates on $(\mathbb{A}^d)^I = (\mathbb{A}^d)^n$:

  $$\begin{cases} z_i = \tilde{z}_i + \tilde{z}_n, & \text{if } i \neq n, \\ z_n = \tilde{z}_n. \end{cases}$$

- $\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)}$: if $\gamma \in \mathcal{A}^{d(n-1),d(n-1)}((\mathbb{A}^d)^{I-\{n\}})$ is a top form, there exists a unique $\gamma' \in C^\infty((\mathbb{A}^d)^{I-\{n\}})$, such that $\gamma = \gamma' \prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)$. We define

  $$\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)}\gamma = \gamma'.$$

  If γ is not a top form, we define

  $$\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)}\gamma = 0.$$

- $d\hat{y}_e^s$: it is an operator defined by

  $$d\hat{y}_e^s = \sum_{j=1}^{n-1}\left(d_{\vec{\Gamma}}^{-1}(t)_{ej} d\bar{\tilde{z}}_j^s + d(d_{\vec{\Gamma}}^{-1}(t)_{ej})(\partial_{\tilde{z}_j^s} + \lambda_j^s)\right),$$

  where $e \in \vec{\Gamma}_1$, $1 \leq s \leq d$.

Proof of propostion 2.15. To simplify notations, we define

$$\tilde{W} = p_{(\vec{\Gamma},m,l)}(-e^{-(z_e|y_e)}dy_e^{m(e)}, -e^{-(z_e|y_e)}(z_e|dy_e)dy_e^{m(e)})e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)} \otimes \beta.$$

We first prove part (1). If $\vec{\Gamma}$ is disconnected, let $\vec{\Gamma}'$ be a connected component of $\vec{\Gamma}$, such that $n \notin \vec{\Gamma}'$. Let $V = \sum\limits_{i \in \vec{\Gamma}'_0} \partial_{\tilde{z}_i}$, ι_V is the interior product operator of V. We can check that

$$\iota_V(\tilde{W}) = 0,$$

so $\tilde{W}$ is not a top form on $(\mathbb{A}^d)^{I-\{n\}}$. Hence

$$W((\vec{\Gamma},m,l), \beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) = \int_{(\mathbb{A}^d)^{I-\{n\}}} \tilde{W} = 0.$$

Now, let's prove (2). If $\vec{\Gamma}$ is connected, by using coordinates $\{\tilde{z}_i\}$, we have

$$\tilde{W} = e^{-\sum\limits_{i=1}^{n-1}\sum\limits_{j=1}^{n-1}(\tilde{z}_i|\bar{\tilde{z}}_j)M_{\vec{\Gamma}}(t)_{ij}} p_{(\vec{\Gamma},m,l)}(-dy_e^{m(e)}, -(\tilde{z}_e|dy_e)dy_e^{m(e)})e^{\sum\limits_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)} \otimes \beta.$$

Since $\vec{\Gamma}$ is connected, $M_{\vec{\Gamma}}(t)$ is invertible by proposition A.7. As our integral over $(\mathbb{A}^d)^{I-\{n\}}$ is of Gaussian type, we can apply Wick's formula to get

$$\begin{aligned}
&W((\vec{\Gamma},m,l),\beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)})\\
&=\int_{(\mathbb{A}^d)^{I-\{n\}}} e^{-\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(\tilde{z}_i|\bar{\tilde{z}}_j)M_{\vec{\Gamma}}(t)_{ij}}\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)\wedge \iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)}\\
&\left(p_{(\vec{\Gamma},m,l)}(-dy_e^{m(e)},-(\tilde{z}_e|dy_e)dy_e^{m(e)})e^{\sum_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum_{i=1}^{n}\lambda_i)}\otimes\beta\right)\\
&=(-2\pi i)^{d(n-1)}\frac{1}{(detM_{\vec{\Gamma}(t)})^d}\left(\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)}\circ e^{\sum_{s=1}^{d}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}M_{\vec{\Gamma}}^{-1}(t)_{ij}\partial_{\tilde{z}_i^s}\partial_{\bar{\tilde{z}}_j^s}}\right.\\
&\left.p_{(\vec{\Gamma},m,l)}(-dy_e^{m(e)},-(\tilde{z}_e|dy_e)dy_e^{m(e)})e^{\sum_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum_{i=1}^{n}\lambda_i)}\otimes\beta\right)\Bigg|_{\tilde{z}_i=0,1\leq i\leq n-1}.
\end{aligned}$$

To simplify the notation further, it is convenient to choose the following basis for one-forms:

$$\begin{cases}
d\tilde{z}_i=d\tilde{z}_i & \text{for } 1\leq i\leq n,\\
d\bar{\tilde{z}}_i=\sum_{j=1}^{n-1}d\left(M_{\vec{\Gamma}(t)_{ij}}^{-1}\bar{z}'_j\right)=\sum_{j=1}^{n-1}\left(d(M_{\vec{\Gamma}(t)_{ij}}^{-1})\bar{z}'_j+M_{\vec{\Gamma}(t)_{ij}}^{-1}d\bar{z}'_j\right) & \text{for } 1\leq i\leq n-1,\\
d\bar{\tilde{z}}_n=d\bar{\tilde{z}}_n & \\
dt_e=dt_e & \text{for } e\in\vec{\Gamma}_1,
\end{cases}$$

where $\bar{z}'_i$ is a smooth vector-valued function defined by

$$\bar{z}'_i=\sum_{j=1}^{n-1}M_{\vec{\Gamma}(t)_{ij}}\bar{z}_j.$$

The dual basis for vectors transforms as follows:

$$\begin{cases}
\partial_{\tilde{z}_i}\to\partial_{\tilde{z}_i} & \text{for } 1\leq i\leq n,\\
\partial_{\bar{\tilde{z}}_i}\to\sum_{j=1}^{n-1}M_{\vec{\Gamma}(t)_{ij}}\partial_{\bar{z}'_j} & \text{for } 1\leq i\leq n-1,\\
\partial_{\bar{\tilde{z}}_n}\to\partial_{\bar{\tilde{z}}_n} & \\
\partial_{t_e}\to\partial_{t_e}-\sum_{i,j,k\in I-\{n\}}M_{\vec{\Gamma}(t)_{ik}}\partial_{t_e}(M_{\vec{\Gamma}(t)_{kj}}^{-1})(\bar{z}'_j|\partial_{\bar{z}'_i}) & \text{for } e\in\vec{\Gamma}_1.
\end{cases}$$

We notice that

$$\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)}=(\det M_{\vec{\Gamma}(t)})^d\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{z}'_i)},$$

so the Feynman graph integrand becomes

$$\begin{aligned}
&W((\vec{\Gamma},m,l),\beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)})\\
&=(-2\pi i)^{d(n-1)}\Bigg(\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{z}'_i)}\circ e^{\sum_{s=1}^{d}\sum_{i=1}^{n-1}\partial_{\tilde{z}_i^s}\partial_{\bar{z}'^s_i}}\\
&p_{(\vec{\Gamma},m,l)}(-dy_e^{m(e)},-(\tilde{z}_e|dy_e)dy_e^{m(e)})e^{\sum_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum_{i=1}^{n}\lambda_i)}\otimes\beta\Bigg)\Bigg|_{\tilde{z}_i=0,1\le i\le n-1}.
\end{aligned}$$

Notice that when applied the exponential, we can trade multiplication by the coordinate function $\tilde{z}_e^s$ with the derivative with respect to $\lambda_e^s=\lambda_{t(e)}^s-\lambda_{h(e)}^s$:

$$\tilde{z}_e^s e^{\sum_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum_{i=1}^{n}\lambda_i)}=\partial_{\lambda_e^s}e^{\sum_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum_{i=1}^{n}\lambda_i)}=(\partial_{\lambda_{t(e)}^s}-\partial_{\lambda_{h(e)}^s})e^{\sum_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum_{i=1}^{n}\lambda_i)},$$

It follows that

$$\begin{aligned}
&p_{(\vec{\Gamma},m,l)}(-dy_e^{m(e)},-(\tilde{z}_e|dy_e)dy_e^{m(e)})e^{\sum_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum_{i=1}^{n}\lambda_i)}\\
&=p_{(\vec{\Gamma},m,l)}(-dy_e^{m(e)},-(\sum_{s=1}^{d}\partial_{\lambda_e^s}\circ dy_e^s)\circ dy_e^{m(e)})e^{\sum_{i=1}^{n-1}(\tilde{z}_i|\lambda_i)+(\tilde{z}_n|\sum_{i=1}^{n}\lambda_i)}.
\end{aligned}$$

We also notice that, as operators, one has

$$e^{\sum_{s=1}^{d}\sum_{i=1}^{n-1}\partial_{\tilde{z}_i^s}\partial_{\bar{z}'^s_i}}\circ dy_e^s=d\tilde{y}_e^s\circ e^{\sum_{s=1}^{d}\sum_{i=1}^{n-1}\partial_{\tilde{z}_i^s}\partial_{\bar{z}'^s_i}},$$

where

$$d\tilde{y}_e^s=\sum_{j=1}^{n-1}\left(d\left(d_{\vec{\Gamma}}^{-1}(t)_{ej}\bar{z}'^s_j\right)+d\left(d_{\vec{\Gamma}}^{-1}(t)_{ej}\right)\partial_{\tilde{z}_j^s}\right)$$

So we have

$$\begin{aligned}
&e^{\sum_{s=1}^{d}\sum_{i=1}^{n-1}\partial_{\tilde{z}_i^s}\partial_{\bar{z}'^s_i}}\circ p_{(\vec{\Gamma},m,l)}(-dy_e^{m(e)},-(\sum_{s=1}^{d}\partial_{\lambda_e^s}\circ dy_e^s)\circ dy_e^{m(e)})\\
&=p_{(\vec{\Gamma},m,l)}(-d\tilde{y}_e^{m(e)},-(\sum_{s=1}^{d}\partial_{\lambda_e^s}\circ d\tilde{y}_e^s)\circ d\tilde{y}_e^{m(e)})\circ e^{\sum_{s=1}^{d}\sum_{i=1}^{n-1}\partial_{\tilde{z}_i^s}\partial_{\bar{z}'^s_i}}.
\end{aligned}$$

Applying these relations, the Feynman graph integrand becomes

$$
\begin{aligned}
&W((\vec{\Gamma}, m, l), \beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) \\
&= (-2\pi i)^{d(n-1)} \Bigg(\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d \bar{z'}_i)} \circ p_{(\vec{\Gamma},m,l)}(-d\tilde{y}_e^{m(e)}, -(\sum_{s=1}^{d} \partial_{\lambda_e^s} \circ d\tilde{y}_e^s) \circ d\tilde{y}_e^{m(e)}) \\
&\circ e^{\sum\limits_{i=1}^{n-1}(\tilde{z}_i|\lambda_i) + (\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)} (1 \otimes \beta) \Bigg)\Bigg|_{\tilde{z}_i=0, 1\le i \le n-1} .
\end{aligned}
$$

The factor $e^{\sum\limits_{s=1}^{d}\sum\limits_{i=1}^{n-1}\partial_{\tilde{z}_i^s}\partial_{\bar{z}'^s_i}}$ disappears since $e^{\sum\limits_{i=1}^{n-1}(\tilde{z}_i|\lambda_i) + (\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)}(1 \otimes \beta)$ is holomorphic. We further notice that

$$
d\tilde{y}_e^s \circ e^{\sum\limits_{i=1}^{n-1}(\tilde{z}_i|\lambda_i) + (\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)} = e^{\sum\limits_{i=1}^{n-1}(\tilde{z}_i|\lambda_i) + (\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)} \circ d\tilde{\tilde{y}}_e^s,
$$

where

$$
d\tilde{\tilde{y}}_e^s = d\tilde{y}_e^s + \sum_{j=1}^{n-1} d\left(d_{\vec{\Gamma}}^{-1}(t)_{ej}\right)\lambda_j^s .
$$

Then we have

$$
\begin{aligned}
&p_{(\vec{\Gamma},m,l)}(-d\tilde{y}_e^{m(e)}, -(\sum_{s=1}^{d} \partial_{\lambda_e^s} \circ d\tilde{y}_e^s) \circ d\tilde{y}_e^{m(e)}) \circ e^{\sum\limits_{i=1}^{n-1}(\tilde{z}_i|\lambda_i) + (\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)} \\
&= e^{\sum\limits_{i=1}^{n-1}(\tilde{z}_i|\lambda_i) + (\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)} \circ p_{(\vec{\Gamma},m,l)}(-d\tilde{\tilde{y}}_e^{m(e)}, -(\sum_{s=1}^{d} (\partial_{\lambda_e^s} + \tilde{\tilde{z}}_e^s) \circ d\tilde{\tilde{y}}_e^s) \circ d\tilde{\tilde{y}}_e^{m(e)}).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&W((\vec{\Gamma}, m, l), \beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) \\
&= (-2\pi i)^{d(n-1)} \Bigg(\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d \bar{z'}_i)} \circ p_{(\vec{\Gamma},m,l)}(-d\tilde{\tilde{y}}_e^{m(e)}, -(\sum_{s=1}^{d} (\partial_{\lambda_e^s} + \tilde{\tilde{z}}_e^s) \circ d\tilde{\tilde{y}}_e^s) \circ d\tilde{\tilde{y}}_e^{m(e)}) \\
&(1 \otimes \beta))|_{\tilde{z}_i=0, 1\le i\le n-1}\, e^{(\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)} \\
&= (-2\pi i)^{d(n-1)} \Bigg(\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d \bar{\tilde{z}}_i)} \circ p_{(\vec{\Gamma},m,l)}(-d\hat{y}_e^{m(e)}, -(\sum_{s=1}^{d} (\partial_{\lambda_e^s} + \tilde{\tilde{z}}_e^s) \circ d\hat{y}_e^s) \circ d\hat{y}_e^{m(e)}) \\
&(1 \otimes \beta))|_{\tilde{z}_i=0, 1\le i\le n-1}\, e^{(\tilde{z}_n|\sum\limits_{i=1}^{n}\lambda_i)} .
\end{aligned}
$$

□

We have shown that the expression $W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)})$ is a well-defined differential form in the variables $t = (t_e) \in (0, +\infty)^{|\Gamma_1|}$. Now, we want to prove that this integrand is integrable over

$$S^+((0, +\infty)^{|\Gamma_1|}) = \{(t_{e_1}, \ldots, t_{e_{|\Gamma_1|}}) \in (0, +\infty)^{|\Gamma_1|} | \sum_{i=1}^{|\Gamma_1|} t_i^2 = 1\}.$$

We use the compactification technique of Schwinger spaces from [20, 19]. We refer to $(0, +\infty)^{\vec{\Gamma}_1}$ as the *Schwinger space* of the graph $\vec{\Gamma}$. In order to integrate, we fix the orientation by the following

$$\int_{(0,L)^{|\Gamma_1|}} \prod_{e \in \vec{\Gamma}_1} dt_e = L^{|\Gamma_1|},$$

where $L > 0$.

There is a natural partial compactification of $(0, +\infty)^{|\Gamma_1|}$, which is constructed by iterated real blow up along corners of $[0, +\infty)^{|\Gamma_1|}$. We collect basic properties of the partially compactified Schwinger spaces in the Appendix B. We use $\widetilde{[0, +\infty)}^{|\Gamma_1|}$ to denote the partially compactified Schwinger space of $\vec{\Gamma}$. One of the key properties of this partial compactification is that the closure of $S^+((0, +\infty)^{|\Gamma_1|})$ in $\widetilde{[0, +\infty)}^{|\Gamma_1|}$, which we denote by

$$\bar{S}^+((0, +\infty)^{|\Gamma_1|}),$$

is compact. Additionally, we will use the fact that the graphical Green's function

$$d_{\vec{\Gamma}}^{-1}(t)_{ei}, \quad i \in \vec{\Gamma}_0 - \{n\}, \quad e \in \vec{\Gamma}_1,$$

which is originally defined on $(0, +\infty)^{|\Gamma_1|}$, can be extended to a smooth function on $\widetilde{[0, +\infty)}^{\vec{\Gamma}_1}$, see lemma B.6 in the appendix.

Proposition 2.16. *Let* $(\vec{\Gamma}, m, l)$ *be a Feynman graph and denote* $\Gamma_0 = I$. *Given*

$$\beta \in (\omega_{\mathbb{A}^d}[d])^{\boxtimes I}$$

and

$$e^{\sum_{i=1}^{n}(z_i|\lambda_i)} = e^{\sum_{s=1}^{d}\sum_{i=1}^{n} z_i^s \lambda_i^s} \in C^\infty((\mathbb{A}^d)^I \times (\mathbb{A}^d)^I),$$

the Feynman graph integrand

$$W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)})$$

can be extended to a smooth differential form on $\widetilde{[0, +\infty)}^{\vec{\Gamma}_1}$. *In particular,*

$$\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)})$$

is convergent.

Proof. By Proposition 2.15, we have an explicit formula for $W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n} (z_i|\lambda_i)})$. Since $d_{\vec{\Gamma}}^{-1}(t)_{ei}$ is smooth on $\widetilde{[0,+\infty)^{\vec{\Gamma}_1}}$, the operator $d\hat{y}_e^s$ is also smooth. So $W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n} (z_i|\lambda_i)})$ can be extended to a smooth differential form. Finally, since $\bar{S}^+((0,+\infty)^{\vec{\Gamma}_1})$ is compact, the integral

$$\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n} (z_i|\lambda_i)}) = \int_{\bar{S}^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n} (z_i|\lambda_i)})$$

is convergent. □

2.2.3. *Properties of Feynman graph integrals.* The next result is about the scaling invariance of the Feynman graph integrals that we are studying.

Proposition 2.17. *Let $(\vec{\Gamma}, m, l)$ be a Feynman graph, such that $\vec{\Gamma}_0 = I$. Given*

$$\beta \in (\omega_{\mathbb{A}^d}[d])^{\boxtimes I}$$

and

$$e^{\sum_{i=1}^{n} (z_i|\lambda_i)} = e^{\sum_{s=1}^{d}\sum_{i=1}^{n} z_i^s \lambda_i^s} \in C^\infty((\mathbb{A}^d)^I \times (\mathbb{A}^d)^I),$$

then we have the following consequences:

(1) *Let $\mathbb{R}_+$ act on Schwinger space by rescaling*

$$\lambda \cdot (t_{e_1}, \ldots, t_{e_{|\Gamma_1|}}) = (\lambda \cdot t_{e_1}, \ldots, \lambda \cdot t_{e_{|\Gamma_1|}}),$$

where $\lambda \in \mathbb{R}_+$, $(t_{e_1}, \ldots, t_{e_{|\Gamma_1|}}) \in (0,+\infty)^{\vec{\Gamma}_1}$. Then, the Feynman graph integrand

$$W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n} (z_i|\lambda_i)})$$

is $\mathbb{R}_+$-invariant and hence induces a smooth differential form on $(0,+\infty)^{\vec{\Gamma}_1}/\mathbb{R}_+$.

(2) *Let $S \subset (0,+\infty)^{\vec{\Gamma}_1}$ be a submanifold, such that the natural map $S \to (0,+\infty)^{\vec{\Gamma}_1}/\mathbb{R}_+$ is a diffeomorphism. Then we have*

$$\int_S W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n} (z_i|\lambda_i)}) = \int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n} (z_i|\lambda_i)}).$$

Proof. Part (2) of the proposition is a direct consequence of (1). Let's prove (1). We need to show that the weight is an equivariant, or basic, differential form with respect to this group action. Let $V = V_1 + V_2$, where $V_1 = \sum_{s=1}^{d}\sum_{i=1}^{n-1} \bar{\tilde{z}}_i^s \partial_{\bar{\tilde{z}}_i^s}$, $V_2 = \sum_{e\in\vec{\Gamma}_1} t_e \partial_{t_e}$. Notice that V_2 is the infinitesimal generaotr for the $\mathbb{R}_+$-action as defined in the statement.

As before, for shorthand notation define

$$\tilde{W} = p_{(\vec{\Gamma},m,l)}(-e^{-(z_e|y_e)}dy_e^{m(e)}, -e^{-(z_e|y_e)}(z_e|dy_e)dy_e^{m(e)})e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)} \otimes \beta.$$

By Proposition 2.9, we have

$$\begin{cases} L_V\tilde{W} = 0, \\ \iota_V\tilde{W} = 0. \end{cases}$$

So we can get

$$L_{V_2}W((\vec{\Gamma},m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) = \int_{(\mathbb{A}^d)^{I-\{n\}}} L_{V_2}\tilde{W} = -\int_{(\mathbb{A}^d)^{I-\{n\}}} L_{V_1}\tilde{W} = 0$$

by Stokes' theorem. Similarly,

$$\iota_{V_2}W((\vec{\Gamma},m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) = \pm\int_{(\mathbb{A}^d)^{I-\{n\}}} \iota_{V_1}\tilde{W} = 0$$

by type reasons. $\square$

We turn to the dependency of the integration $\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma},m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)})$ on the actual Feynman graph $(\vec{\Gamma},m,l)$. This is summarized in the following result.

Proposition 2.18. *Let $(\vec{\Gamma},m,l)$ be a Feynman graph, such that $\vec{\Gamma}_0 = I$. Given*

$$\beta \in (\omega_{\mathbb{A}^d}[d])^{\boxtimes I}$$

and

$$e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)} = e^{\sum\limits_{s=1}^{d}\sum\limits_{i=1}^{n} z_i^s\lambda_i^s} \in C^\infty((\mathbb{A}^d)^I \times (\mathbb{A}^d)^I),$$

then we have the following consequences:

(1) *Let $1 \leq l_1, l_2 < l$, $l \leq l_1', l_2' \leq |\Gamma_1|$, we use $\sigma_{l_1l_2}(\vec{\Gamma})$ ($\sigma_{l_1'l_2'}(\vec{\Gamma})$) to denote the Feynman graph $\vec{\Gamma}$ with the order of $e_{l_1}, e_{l_2} \in \vec{\Gamma}_1$ ($e_{l_1'}, e_{l_2'} \in \vec{\Gamma}_1$) interchanged. Then we have*

$$\begin{cases} \int_{S^+((0,+\infty)^{\sigma_{l_1l_2}(\vec{\Gamma}_1)})} W((\sigma_{l_1l_2}(\vec{\Gamma}),m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) = \int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma},m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) \\ \int_{S^+((0,+\infty)^{\sigma_{l_1'l_2'}(\vec{\Gamma}_1)})} W((\sigma_{l_1'l_2'}(\vec{\Gamma}),m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) = -\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma},m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}) \end{cases}$$

(2) *For $1 \leq s \leq d$, $i, j \in \vec{\Gamma}_0$, let $(\vec{\Gamma}', m^s, l)$ be the Feynman graph which satisfies the following:*
  - *$\vec{\Gamma} \subset \vec{\Gamma}'$ is a directed subgraph.*
  - *$\vec{\Gamma}_0' = \vec{\Gamma}_0$.*
  - *$\vec{\Gamma}_1' = \{e_0\} \cup \vec{\Gamma}_1$, where $t(e_0) = i$, $h(e_0) = j$.*

- *$m(e_0) = s$.*

Then we have

$$\sum_{s=1}^{d} \int_{S^+((0,+\infty)^{\vec{\Gamma}'_1})} W((\vec{\Gamma}', m^s, l), z^s_{ij} \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)}) = \int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)}). \tag{2.5}$$

(3) *For $1 \leq s \leq d$, $i, j \in \vec{\Gamma}_0$, let $(\vec{\Gamma}'', m^s, l)$ be the Feynman graph which satisfies the following:*
- *$\vec{\Gamma} \subset \vec{\Gamma}''$ is a directed subgraph.*
- *$\vec{\Gamma}''_0 = \vec{\Gamma}_0$.*
- *$\vec{\Gamma}''_1 = \vec{\Gamma}_1 \cup \{e_{|\vec{\Gamma}_1|+1}\}$, where $t(e_{|\vec{\Gamma}_1|+1}) = i$, $h(e_{|\vec{\Gamma}_1|+1}) = j$.*
- *$m(e_{|\vec{\Gamma}_1|+1}) = s$.*

Then we have

$$\sum_{s=1}^{d} \int_{S^+((0,+\infty)^{\vec{\Gamma}''_1})} W((\vec{\Gamma}'', m^s, l), z^s_{ij} \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)}) = 0.$$

Proof. Part (1) is a direct observation. We prove part (2). Let's first assume $\vec{\Gamma}_1 \neq \emptyset$. To simplify notations, we define

$$\tilde{W} = p_{(\vec{\Gamma}, m, l)}(-e^{-(z_e|y_e)} dy_e^{m(e)}, -e^{-(z_e|y_e)}(z_e|dy_e) dy_e^{m(e)}) e^{\sum_{i=1}^{n}(z_i|\lambda_i)} \otimes \beta.$$

Let's consider a submanifold

$$S = S^+((0,+\infty)^{\vec{\Gamma}_1}) \times (0,+\infty) \subset S^+((0,+\infty)^{\vec{\Gamma}'_1}).$$

We notice that the natural map from S to $(0,+\infty)^{\vec{\Gamma}'_1}/\mathbb{R}_+$ is a diffeomophism, so

$$\begin{aligned}
&\sum_{s=1}^{d} \int_{S^+((0,+\infty)^{\vec{\Gamma}'_1})} W((\vec{\Gamma}', m^s, l), z^s_{ij} \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)}) \\
&= \sum_{s=1}^{d} \int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} \int_{(0,+\infty)} W((\vec{\Gamma}', m^s, l), z^s_{ij} \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)}) \\
&= \sum_{s=1}^{d} \int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} \int_{(\mathbb{A}^d)^{I-\{n\}}} z^s_{ij} x^s_{ij} \tilde{W} \\
&= \int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)}).
\end{aligned}$$

When $\vec{\Gamma}_1=\emptyset$, $|\Gamma_0|\geq 3$, $\vec{\Gamma}$ and $\vec{\Gamma}'$ are disconnected, so both sides of formula (2.5) is zero. When $\vec{\Gamma}_1=\emptyset$, $i=1$, $j=2$, $\vec{\Gamma}_0=\{1,2\}$, we have

$$\begin{aligned}
&\sum_{s=1}^{d}\int_{S^+((0,+\infty)^{\vec{\Gamma}'_1})} W((\vec{\Gamma}',m^s,l),z_{ij}^s\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)})\\
&=\left(-\int_{\mathbb{A}^d} e^{-(z_{12}|y_{12})}(z_{12}|dy_{12})\beta e^{(z_1|\lambda_1)+(z_2|\lambda_2)}\right)\Bigg|_{t_{12}=1}\\
&=\left(\int_{\mathbb{A}^d}\bar{\partial}\left(e^{-(z_{12}|y_{12})}\beta e^{(z_1|\lambda_1)+(z_2|\lambda_2)}\right)\right)\Bigg|_{t_{12}=1}\\
&=0.
\end{aligned}$$

Since $\vec{\Gamma}$ is disconnected,

$$\begin{aligned}
&\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} W((\vec{\Gamma},m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)})\\
&=0\\
&=\sum_{s=1}^{d}\int_{S^+((0,+\infty)^{\vec{\Gamma}'_1})} W((\vec{\Gamma}',m^s,l),z_{ij}^s\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}).
\end{aligned}$$

Finally, part (3) can be proved by similar arguments. □

2.3. **Higher residues from Feynman graphs.** Finally, we arrive at the definition of the residue on the model $\mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I})$ for configuration space.

Definition 2.19. Let

$$\alpha=p(x_{ij}^s,\mathbf{d}x_{ij}^s)\otimes\beta\in\mathbf{J}_I^{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I}),$$

where $p(x_{ij}^s,\mathbf{d}x_{ij}^s)$ is a monomial with coefficient 1. Let

$$e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}=e^{\sum\limits_{s=1}^{d}\sum\limits_{i=1}^{n}z_i^s\lambda_i^s}\in C^\infty((\mathbb{A}^d)^I\times(\mathbb{A}^d)^I),$$

then the *residue* of $\alpha e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}$ is defined by

$$\begin{aligned}
&\frac{-1}{(-2\pi i)^{d(n-1)}}\oint_{z_1,\ldots z_{n-1}=z_n}\alpha e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}\\
:=&\frac{(-1)^{\frac{1}{2}(|\Gamma_1|-l)(\Gamma_1|-l+1)+1}}{(-2\pi i)^{d(n-1)}}\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})}W((\vec{\Gamma},m,l),\beta e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}),
\end{aligned}$$

where $(\vec{\Gamma},m,l)$ is a Feynman graph with corresponding polynomial $p_{(\vec{\Gamma},m,l)}=p(x_{ij}^s,\mathbf{d}x_{ij}^s)$, see definition 2.11. We define residues for general elements in $\mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I})$ by linearity.

We observe that by proposition 2.18, residues are well-defined. The following proposition highlights key features of our higher residue defined on the Jouanolou model. In part, it implies that the formal $\mathcal{D}_{(\mathbb{A}^d)^I}$-module structure on $\mathbf{J}^I_{\mathbb{A}^d}$ coincides with the action by derivatives on the residues we have just defined.

Proposition 2.20. *Let* $\alpha \in \mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I})$, $e^{\sum_{i=1}^{n}(z_i|\lambda_i)} \in C^\infty((\mathbb{A}^d)^I \times (\mathbb{A}^d)^I)$, *and* $1 \le s \le d$. *We have*

(1) *If* $i \in I - \{n\}$, *then*

$$\frac{-1}{(-2\pi i)^{d(n-1)}} \oint_{z_1,\ldots z_{n-1}=z_n} \left(\alpha e^{\sum_{i=1}^{n}(z_i|\lambda_i)}\right) \cdot \partial_{z_i^s} = 0.$$

(2) *If* $i = n$, *then*

$$\begin{aligned}&\frac{-1}{(-2\pi i)^{d(n-1)}} \oint_{z_1,\ldots z_{n-1}=z_n} \left(\alpha e^{\sum_{i=1}^{n}(z_i|\lambda_i)}\right) \cdot \partial_{z_i^s} \\ &= \left(\frac{-1}{(-2\pi i)^{d(n-1)}} \oint_{z_1,\ldots z_{n-1}=z_n} \alpha e^{\sum_{i=1}^{n}(z_i|\lambda_i)}\right) \cdot \partial_{z_n^s}.\end{aligned}$$

Proof. Without loss of generality, we assume $\alpha = p_{(\vec{\Gamma},m,l)} \otimes \beta$, where $(\vec{\Gamma}, m, l)$ is a Feynman graph. We can also assume $\vec{\Gamma}_1 \neq \emptyset$, $t(e_1) = i$.

Let $\vec{\Gamma}' \subset \vec{\Gamma}$ be a subgraph, such that $\vec{\Gamma}'_0 = \vec{\Gamma}_0$, $\vec{\Gamma}'_1 = \vec{\Gamma}_1 - \{e_1\}$. Let

$$\tilde{W}' = p_{(\vec{\Gamma}',m|_{\vec{\Gamma}'},l)}(-e^{-(z_e|y_e)}dy_e^{m(e)}, -e^{-(z_e|y_e)}(z_e|dy_e)dy_e^{m(e)})e^{\sum_{i=1}^{n}(z_i|\lambda_i)} \otimes \beta.$$

When $1 < l$, we have

$$\partial_{z_i^s} x_{e_1}^{m(e_1)} = -x_{e_1}^s x_{e_1}^{m(e_1)}.$$

Let's prove

$$\begin{aligned}&\int_{S^+((0,+\infty)^{|\Gamma_1|+1})} \int_{(\mathbb{A}^d)^{I-\{n\}}} e^{-\frac{(z_{e_1}|\bar{z}_{e_1})}{t_{e_0}}} d(\frac{\bar{z}_{e_1}^s}{t_{e_0}}) e^{-\frac{(z_{e_1}|\bar{z}_{e_1})}{t_{e_1}}} d(\frac{\bar{z}_{e_1}^{m(e_1)}}{t_{e_1}}) \tilde{W}' \\ &= -\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} \int_{(\mathbb{A}^d)^{I-\{n\}}} \partial_{z_i^s} \left(e^{-\frac{(z_{e_1}|\bar{z}_{e_1})}{t_{e_1}}} d(\frac{\bar{z}_{e_1}^{m(e_1)}}{t_{e_1}})\right) \tilde{W}'.\end{aligned}$$

When $\vec{\Gamma}'_1 \neq \emptyset$, let

$$S = S^+((0,+\infty)^{\vec{\Gamma}'_1}) \times (0,+\infty)^2 \subset (0,+\infty)^{|\Gamma_1|+1}.$$

Note the natural map from S to $(0,+\infty)^{|\Gamma_1|+1}/\mathbb{R}_+$ is a diffeomorphism, we have

$$
\begin{aligned}
&\int_{S^+((0,+\infty)^{|\Gamma_1|+1})}\int_{(\mathbb{A}^d)^{I-\{n\}}} e^{-\frac{(z_{e_1}|\bar{z}_{e_1})}{t_{e_0}}} d(\frac{\bar{z}^s_{e_1}}{t_{e_0}}) e^{-\frac{(z_{e_1}|\bar{z}_{e_1})}{t_{e_1}}} d(\frac{\bar{z}^{m(e_1)}_{e_1}}{t_{e_1}})\tilde{W}' \\
&= \int_{S^+((0,+\infty)^{\vec{\Gamma}'_1})}\int_{(\mathbb{A}^d)^{I-\{n\}}} i(x^s_{e_1}) i(x^{m(e_1)}_{e_1})\tilde{W}' \\
&= -\int_{S^+((0,+\infty)^{\vec{\Gamma}'_1})}\int_{(\mathbb{A}^d)^{I-\{n\}}} \partial_{z^s_i}(i(x^{m(e_1)}_{e_1}))\tilde{W}' \\
&= -\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})}\int_{(\mathbb{A}^d)^{I-\{n\}}} \partial_{z^s_i}\left(e^{-\frac{(z_{e_1}|\bar{z}_{e_1})}{t_{e_1}}} d(\frac{\bar{z}^{m(e_1)}_{e_1}}{t_{e_1}})\right)\tilde{W}'.
\end{aligned}
$$

When $\vec{\Gamma}'_1=\emptyset$, this can be proved by directed computation.

We can use similar arguments to prove the case when $l=1$.

Now, (1) follows from the fact that the integral of a total derivative is zero. By dominated convergence theorem for derivatives, we can interchange the order of taking derivative and integration. So we proved (2). $\square$

2.4. **Construction of the unit chiral algebra on $\mathbb{A}^d$.**

We culminate this section with a construction of an L_∞ chiral algebra structure on $\omega_{\mathbb{A}^d}[d-1]$ by using residues. This is precisely the *unit chiral algebra* in dimension d.

We first introduce the following definition.

Definition 2.21. Let $\alpha\in\mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I})$, the shifted n operation

$$\tilde{\mu}_I:\mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I})\to\omega_{\mathbb{A}^d}[d]\otimes_{k[\lambda_\star]}k[\lambda_1,\dots,\lambda_n]$$

is given by

$$\tilde{\mu}_I(\alpha)=\frac{-e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}}\left(\oint_{z_1,\cdots z_{n-1}=z_n}\alpha e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)}\right)\Bigg|_{z_n=w},$$

where $\lambda_\star=\sum\limits_{i=1}^{n}\lambda_i$.

Remark 2.22. By proposition 2.15, we know $\tilde{\mu}_I(\alpha)$ is a well-defined element of $\omega_{\mathbb{A}^d}[d]\otimes_{k[\lambda_\star]}k[\lambda_1,\dots,\lambda_n]$, i.e., $\tilde{\mu}_I(\alpha)$ is a polynomial with respect to $\{\lambda_i\}_{i\in I}$.

As mentioned in Remark 1.11, the Jouanolou model has the advantage that it carries a GL_d-action. Here we give the detailed definition.

Definition 2.23. The Jouanolou model $\mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d])^{\boxtimes I})$ and the push forward $\omega_{\mathbb{A}^d}[d]\otimes_{k[\lambda_\bullet]}k[\lambda_1,\dots,\lambda_n]$ carry a natural $\mathrm{GL_d}$ action which can be described as follows

$$(z^1_i,\dots,z^d_i)\mapsto(z^1_i,\dots,z^d_i)A,\quad i=1,\dots,n,$$

$$(dz_i^1, \ldots, dz_i^d) \mapsto (dz_i^1, \ldots, dz_i^d)A, \quad i = 1, \ldots, n,$$

$$(x_{ij}^1, \ldots, x_{ij}^d) \mapsto (x_{ij}^1, \ldots, x_{ij}^d)(A^T)^{-1}, \quad i, j = 1, \ldots, n,$$

$$(\lambda_i^1, \ldots, \lambda_i^d) \mapsto (\lambda_i^1, \ldots, \lambda_i^d)(A^T)^{-1}, \quad i = \star, 1, \ldots, n,$$

where $A \in \mathrm{GL}_d$.

The following proposition says that $\tilde{\mu}_I(\alpha)$ defines a degree 1 GL_d-equivariant chiral operation.

Proposition 2.24. *$\tilde{\mu}_I(\alpha)$ defines a cohomological degree 1 element of*

$$\mathrm{Hom}_{\mathcal{D}_{(\mathbb{A}^d)^I}}\left(\mathbf{J}^I_{\mathbb{A}^d}((\omega_{\mathbb{A}^d}[d]))^{\boxtimes I}), \Delta_*^{I/\{\star\}}(\omega_{\mathbb{A}^d}[d])\right)$$

which is furthermore GL_d-equivariant.

Proof. We first prove that $\tilde{\mu}_I$ is a $\mathcal{D}_{(\mathbb{A}^d)^I}$-module map. First, observe that functions act in the following way:

$$\begin{aligned}
&\tilde{\mu}_I(\alpha \cdot z_i^s) \\
&= \frac{-e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1, \cdots z_{n-1} = z_n} z_i^s \alpha e^{\sum\limits_{j=1}^{n}(z_j|\lambda_j)} \right)\Bigg|_{z_n = w} \\
&= \frac{-e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1, \cdots z_{n-1} = z_n} \alpha \partial_{\lambda_i^s} e^{\sum\limits_{j=1}^{n}(z_j|\lambda_j)} \right)\Bigg|_{z_n = w} \\
&= (w + \partial_{\lambda_i^s}) \frac{-e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1, \cdots z_{n-1} = z_n} \alpha e^{\sum\limits_{j=1}^{n}(z_j|\lambda_j)} \right)\Bigg|_{z_n = w} \\
&= \tilde{\mu}_I(\alpha) \cdot z_i^s,
\end{aligned}$$

Similarly, derivatives act as

$$
\begin{aligned}
&\tilde{\mu}_I(\alpha \cdot \partial_{z_i^s}) \\
&= \frac{e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1,\cdots z_{n-1}=z_n} \partial_{z_i^s}(\alpha) e^{\sum_{j=1}^{n}(z_j|\lambda_j)} \right)\Bigg|_{z_n=w} \\
&= \frac{e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1,\cdots z_{n-1}=z_n} \partial_{z_i^s}(\alpha e^{\sum_{j=1}^{n}(z_j|\lambda_j)}) \right)\Bigg|_{z_n=w} \\
&- \frac{e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1,\cdots z_{n-1}=z_n} \lambda_i^s \alpha e^{\sum_{j=1}^{n}(z_j|\lambda_j)} \right)\Bigg|_{z_n=w} \\
&= \frac{e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1,\cdots z_{n-1}=z_n} \partial_{z_i^s}(\alpha e^{\sum_{j=1}^{n}(z_j|\lambda_j)}) \right)\Bigg|_{z_n=w} + \tilde{\mu}_I(\alpha) \cdot \partial_{\lambda_i^s}.
\end{aligned}
$$

Using proposition 2.20, the first term is 0 when $i \neq n$. When $i = n$, the first term is

$$
(-\lambda_\star + \partial_w) \frac{e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1,\cdots z_{n-1}=z_n} \alpha e^{\sum_{j=1}^{n}(z_j|\lambda_j)} \right)\Bigg|_{z_n=w},
$$

which is zero as an element in

$$
\omega_{\mathbb{A}^d}[d] \otimes_{k[\lambda_\star]} k[\lambda_1, \ldots, \lambda_n].
$$

We next check that $\tilde{\mu}$ is of cohomological degree 1. Assume

$$
\alpha = p \otimes \beta \in \mathbf{J}^I_{\mathbb{A}^d} \otimes_{\mathcal{O}_{(\mathbb{A}^d)^I}} (\omega_{\mathbb{A}^d}[d])^{\boxtimes I},
$$

we use $|p|$ and $|\beta|$ to denote the homological degree of p and β respectively. To make sure the integral

$$
\oint_{z_1,\cdots z_{n-1}=z_n} p \otimes \beta e^{\sum_{j=1}^{n}(z_j|\lambda_j)}
$$

is non-zero, we have

$$
\begin{cases} |\Gamma_1| + |p| = d(|\Gamma_0| - 1) + |\Gamma_1| - 1 \\ |\beta| = -d|\Gamma_0| \end{cases}.
$$

So $|\tilde{\mu}_I(\alpha)| = -d = |p| + |\beta| + 1 = |\alpha| + 1$.

For the GL_d-equivariance, we use Proposition 2.15 (2). In that explicit formula, we can formally treat $\tilde{z}_i$ and $\bar{\tilde{z}}_i$ as independent variables and declare that under the action $A \in \mathrm{GL}_d$

$$
(\tilde{z}_i^1, \ldots, \tilde{z}_i^d) \mapsto (\tilde{z}_i^1, \ldots, \tilde{z}_i^d)A, \quad i = 1, \ldots, n,
$$

$$(\bar{\tilde{z}}_i^1, \dots, \bar{\tilde{z}}_i^d) \mapsto (\bar{\tilde{z}}_i^1, \dots, \bar{\tilde{z}}_i^d)(A^T)^{-1}, \quad i = 1, \dots, n.$$

Then in the expression of $W((\vec{\Gamma}, m, l), \beta e^{\sum_{i=1}^{n}(z_i|\lambda_i)})$, we see that both of $\iota_{\prod_{i=1}^{n-1}(d^d\tilde{z}_i d^d\bar{\tilde{z}}_i)}$ and $\sum_{s=1}^{d}(\partial_{\lambda_e^s} + \tilde{z}_e^s) \circ d\hat{y}_e^s$ are invariant and for $e \in \Gamma_1$

$$(d\hat{y}_e^1, \dots, d\hat{y}_e^d) \mapsto (d\hat{y}_e^1, \dots, d\hat{y}_e^d)(A^T)^{-1}$$

has the expected transformation rule. □

Thus, $\{\tilde{\mu}_I\}$ defines a chiral operation of the appropriate degree. In order for such operations to define a chiral algebra, we need to check that $\tilde{\mu}_I$ satisfy the following shifted L_∞ relations. This is the main result of this section.

Theorem 2.25. *$\{\tilde{\mu}_I\}$ satisfies the following properties:*

(1) *$\tilde{\mu}_I = \tilde{\mu}_{I_{\sigma_{ii'}}} \circ \sigma_{ii'}$, where $\sigma_{ii'}$ is the permutation of $i \neq i' \in I$ (recall it is defined in Definition 1.6).*

(2) *For any finite set I one has (recall the notation in the proof of Theorem 1.7)*

$$-\tilde{\mu}_I \circ \mathbf{d} = \sum_{I' \subset I} \tilde{\mu}_{\{\bullet\} \cup I - I'} \circ \tilde{\mu}_{I' \subset I}$$

where the sum is over subsets $I' \subset I$.

Proof. Let's prove (1) first. When $i, i' \neq n$, it is trivial. Let's assume $i < i' = n$. Since $\tilde{\mu}_I$ is a $\mathcal{D}_{(\mathbb{A}^d)^I}$-module morphism, we only need to prove this in the case when

$$\alpha = p(x_i^s, \mathbf{d}x_i^s) \otimes \prod_{i=1}^{n} d^d z_i.$$

In this case,

$$\tilde{\mu}_I(\alpha) = \frac{-e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1, \cdots z_{n-1} = z_n} \alpha e^{\sum_{j=1}^{n}(z_j|\lambda_j)} \right) \Bigg|_{z_n = w}.$$

By using the coordinate transform

$$z_j = z'_j + 2w - z'_n - z'_i,$$

we have

$$\tilde{\mu}_I(\alpha) = \frac{-e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z'_1, \cdots, z'_{i-1}, z'_n, z'_{i+1}, \cdots, z'_{n-1} = z'_i} (-1)^{d^2} \alpha e^{\sum_{j=1}^{n}(z'_j|\lambda_j) + (\lambda_\star|(z'_i - z'_n)}) \right) \Bigg|_{z'_i = w}.$$

By Proposition 2.15 and the fact that $d^d w \cdot \lambda_\star^s = 0$, we have

$$\tilde{\mu}_I(\alpha) = \frac{-e^{-(\lambda_\star|w)}}{(-2\pi i)^{d(n-1)}} \left(\oint_{z_1',\cdots,z_{i-1}',z_n',z_{i+1}',\cdots,z_{n-1}'=z_i'} \sigma_{in}(\alpha) e^{\sum\limits_{j=1}^{n}(z_j'|\lambda_j)} \right)\Bigg|_{z_i'=w} = \tilde{\mu}_{I_{\sigma_{in}}} \circ \sigma_{in}(\alpha).$$

Now, let's prove (2). Assume $\alpha = p_{(\vec{\Gamma},m,l)} \otimes \beta$, where $(\vec{\Gamma}, m, l)$ is a Feynman graph. Let

$$\tilde{W} = p_{(\vec{\Gamma},m,l)}(-e^{-(z_e|y_e)} dy_e^{m(e)}, -e^{-(z_e|y_e)}(z_e|dy_e) dy_e^{m(e)}) e^{\sum\limits_{i=1}^{n}(z_i|\lambda_i)} \otimes \beta.$$

By Stokes' theorem and Proposition B.7,

$$\begin{aligned}
&-\tilde{\mu}_I(\mathbf{d}\alpha)\\
&= \frac{e^{-(\lambda_\star|w)}(-1)^{\frac{1}{2}(|\Gamma_1|-l)(\Gamma_1|-l+1)}}{(-2\pi i)^{d(n-1)}} \left(\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} \int_{(\mathbb{A}^d)^{I-\{n\}}} (\bar{\partial} + d_t)\tilde{W} \right)\Bigg|_{z_n=w}\\
&= \frac{e^{-(\lambda_\star|w)}(-1)^{\frac{1}{2}(|\Gamma_1|-l)(\Gamma_1|-l+1)}}{(-2\pi i)^{d(n-1)}} \left(\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})} d_t \left(\int_{(\mathbb{A}^d)^{I-\{n\}}} \tilde{W} \right) \right)\Bigg|_{z_n=w}\\
&= \frac{e^{-(\lambda_\star|w)}(-1)^{\frac{1}{2}(|\Gamma_1|-l)(\Gamma_1|-l+1)}}{(-2\pi i)^{d(n-1)}} \left(\sum_{\vec{\Gamma}'\subset\vec{\Gamma}} \operatorname{sgn}(\sigma_{\vec{\Gamma}_1'\subset\vec{\Gamma}_1}) \int_{S^+((0,+\infty)^{\vec{\Gamma}_1-\vec{\Gamma}_1'})} \int_{S^+((0,+\infty)^{\vec{\Gamma}_1'})} \right.\\
&\left. \int_{(\mathbb{A}^d)^{\{\bullet\}\cup\vec{\Gamma}_0-\vec{\Gamma}_0'-\{n\}}} \int_{(\mathbb{A}^d)^{|\Gamma_0'|-1}} \tilde{W} \right)\Bigg|_{z_n=w}\\
&= \sum_{I'\subset I} \tilde{\mu}_{\{\bullet\}\cup I-I'} \circ \tilde{\mu}_{I'\subset I}(\alpha).
\end{aligned}$$

□

Finally, using the canonical décalage isomorphism

$$\mathrm{Sym}^{\boxtimes I}(\omega_{\mathbb{A}^d}[d]) \cong \wedge^{\boxtimes I}(\omega_{\mathbb{A}^d}[d-1])[n],$$

the shifted L_∞ chiral algebra structure $\{\tilde{\mu}_I\}_{I\in\mathbf{fSet}}$ on $\omega_{\mathbb{A}^d}[d]$ corresponds to an L_∞ chiral algebra structure on $\omega_{\mathbb{A}^d}[d-1]$.

Definition 2.26. The L_∞ algebra structure on $\omega_{\mathbb{A}^d}[d-1]$ is called the *unit* L_∞ chiral algebra structure. We denote it by $\omega_{\mathbb{A}^d}^{\blacklozenge}$. The corresponding L_∞ operations are denoted by $\{\mu_I^\omega\}_{I\in\mathbf{fSet}}$.

3. EXAMPLES OF HIGHER CHIRAL ALGEBRAS

We provide examples of chiral algebras in the Jouanolou model. These include commutative chiral algebras and free ghost chiral algebras. Additionally, we provide a construction of a chiral algebra enhancement of the higher-dimensional Kac–Moody algebra as constructed in [5]. This is generalization of the relationship between the (universal) WZW/current chiral algebras and affine Kac–Moody algebras. We will assume that $\mathbf{k} = \mathbb{C}$ in this section. However, everything works for the general ground field $\mathbf{k}$ if one accepts the existence of a unit chiral algebra over $\mathbf{k}$.

For simplicity of notation, we write ω instead of $\omega_{\mathbb{A}^d}$ throughout this section.

3.1. Commutative chiral algebras.

Following [2], we define the notion of commutative chiral algebras.

Definition 3.1. Let $\mathcal{A}$ be a chiral algebra with zero differential on $\mathbb{A}^d$. We say that $\mathcal{A}$ is *commutative* if the composition

$$\Gamma\left((\mathbb{A}^d)^I, \mathcal{A}^{\boxtimes I}\right) \hookrightarrow \mathbf{J}^I_{\mathbb{A}^d}(\mathcal{A}) \xrightarrow{\mu_I} \mathcal{A} \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I]$$

vanishes for all I with $|I| \geq 2$. Here, the first map is the inclusion of global sections into derived global sections.

A graded commutative $\mathcal{D}_{\mathbb{A}^d}$-algebra $\mathbf{B}$ is a left $\mathcal{D}_{\mathbb{A}^d}$-module with a commutative product

$$m \colon \mathbf{B} \otimes \mathbf{B} \to \mathbf{B}$$

which is a $\mathcal{D}_{\mathbb{A}^d}$-module map.

Proposition 3.2. *Let* $\mathbf{B}$ *be a commutative* $\mathcal{D}_{\mathbb{A}^d}$*-algebra. There is a chiral algebra structure on* $\mathcal{B} := \mathbf{B} \otimes_{\mathcal{O}_{\mathbb{A}^d}} \omega^{\blacklozenge}_{\mathbb{A}^d}$ *which makes* $\mathcal{B}$ *into a commutative chiral algebra.*

Proof. We define

$$\mu^{\mathcal{B}}_I : \mathbf{J}^I_{\mathbb{A}^d}(\mathcal{B}^{\boxtimes I}) \to \mathcal{B} \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I]$$

to be the composition of the following sequence of maps

$$\begin{aligned}\mathbf{J}^I_{\mathbb{A}^d}(\mathcal{B}^{\boxtimes I}) = \mathbf{J}^I_{\mathbb{A}^d}((\omega^{\blacklozenge}_{\mathbb{A}^d})^{\boxtimes I}) \otimes_{\mathcal{O}_{(\mathbb{A}^d)^I}} \mathbf{B}^{\boxtimes I} \xrightarrow{\mu^{\omega}_I \otimes \mathrm{Id}} \left(\omega^{\blacklozenge}_{\mathbb{A}^d} \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I]\right) \otimes_{\mathcal{O}_{(\mathbb{A}^d)^I}} \mathbf{B}^{\boxtimes I} \\ \xrightarrow{\sim} \omega^{\blacklozenge}_{\mathbb{A}^d} \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I] \otimes_{\mathbf{k}[\lambda_I]} \left(\mathbf{B}^{\boxtimes I} \otimes \mathbf{k}[\lambda_I]\right) \xrightarrow{\sim} \omega^{\blacklozenge}_{\mathbb{A}^d} \otimes_{\mathbf{k}[\lambda_\star]} \left(\mathbf{B}^{\otimes I} \otimes \mathbf{k}[\lambda_I]\right) \\ \xrightarrow{m^{(I)}} \omega^{\blacklozenge}_{\mathbb{A}^d} \otimes_{\mathbf{k}[\lambda_\star]} (\mathbf{B} \otimes \mathbf{k}[\lambda_I]) \xrightarrow{\sim} \mathcal{B} \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I].\end{aligned}$$

Here $m^{(I)}$ is the iterated commutative product. The fact that $\{\mu^{\mathcal{B}}_I\}$ satisfy the L_∞ relations follows immediately from the L_∞-relations of the unit chiral algebra. ☐

3.2. **Free ghost chiral algebras.** The following set of examples are immediate generalizations of familiar $d = 1$ ghost-type chiral algebras (of $\beta - \gamma$ or $b - c$ type). Let F be sheaf of locally free $\mathbb{Z}$-graded $\mathcal{O}_{\mathbb{A}^d}$-modules and $F_{\mathcal{D}} = F \otimes_{\mathcal{O}} \mathcal{D}$ the induced dg $\mathcal{D}_{\mathbb{A}^d}$-module. We assume that F is equipped with a non-degenerate, graded skew-symmetric, pairing of graded $\mathcal{O}$-modules

$$\langle -,- \rangle \colon F \otimes_{\mathcal{O}} F \to \omega[d-1]. \tag{3.1}$$

We introduce the following graded commutative $\mathcal{D}$-algebra

$$\mathbf{F} = \mathrm{Sym}\left(F_{\mathcal{D}} \otimes \omega^{-1}\right)$$

Remark 3.3. In the case that $F = \omega_{\mathbb{A}^d} \oplus \mathcal{O}[1-d]$, with the obvious pairing, the chiral algebra we are about to define is associated to the following first-order action functional in physics

$$\int_{\mathbb{C}^d} \beta \bar{\partial} \gamma \tag{3.2}$$

where β is a form of Dolbeault type $(d, d-1)$ and γ is a smooth function. Indeed, when $d = 1$ this returns the ordinary $\beta - \gamma$ system used in superstring theory. To obtain the $b - c$ system we must introduce an additional $\mathbb{Z}/2$ grading and consider the parity shift of this example. The same generalization can be made in the higher-dimensional setting as well, though we will not pursue it here.

Definition 3.4. Define the $\mathcal{D}$-module map

$$e^{\mathcal{P}_{12}} \colon \mathbf{J}^{\{1,2\}}_{\mathbb{A}^d}(\mathbf{F}^{\boxtimes 2}) \to \mathbf{J}^{\{1,2\}}_{\mathbb{A}^d}(\mathbf{F}^{\boxtimes 2})$$

by extending the $\langle -,- \rangle$ pairing

$$\mathcal{P}_{12} = \langle -,- \rangle \cdot P_{12} \colon (F \otimes \omega^{-1}) \boxtimes F[1-d] \to \mathbf{J}^{\{1,2\}}_{\mathbb{A}^d}(\mathcal{O} \boxtimes \mathcal{O})$$

by the Leibniz rule. Here we define the element

$$P_{12} = \sum_{s=1}^{d} (-1)^{s-1} x_{12}^s \cdot \mathbf{d}x_{12}^1 \cdots \widehat{\mathbf{d}x_{12}^s} \cdots \mathbf{d}x_{12}^d,$$

which under the map $x_{12}^s \mapsto \frac{\bar{z}_1^s - \bar{z}_2^s}{|z_1 - z_2|^2}$ is exactly the Bochner-Martinelli kernel.

Denote

$$\mathcal{P}_I = \sum_{\{i,j\} \subset I} \mathcal{P}_{\{i,j\}}.$$

If $I = I' \bigsqcup I''$, we write $\mathcal{P}_I = \sum_{i' \in I', i'' \in I''} \mathcal{P}_{\{i', i''\}}$.

Definition 3.5. We define the chiral operations $\{\mu_I^{\mathcal{F}}\}$ on $\mathcal{F} = \mathbf{F} \otimes_{\mathcal{O}_{\mathbb{A}^d}} \omega^{\blacklozenge}_{\mathbb{A}^d}$ as follows. Let $\{\mu_I^{\mathcal{F},comm}\}$ be the commutative chiral operations as constructed in proposition 3.2. Then, define the new chiral operations $\mu_I^{\mathcal{F}}$ by the composition:

$$\mu_I^{\mathcal{F}} = \mu_I^{\mathcal{F},comm} \circ e^{\mathcal{P}_I} \colon \mathbf{J}^I_{\mathbb{A}^d}\left(\mathcal{A}^{\boxtimes I}\right) \to \mathcal{A} \otimes_{\mathbf{k}[\lambda_\star]} \mathbf{k}[\lambda_I].$$

We have the following lemma.

Lemma 3.6. *Let* $\alpha_{I'} \in \mathbf{J}^{I'}_{\mathbb{A}^d}((\omega^{\blacklozenge}_{\mathbb{A}^d})^{\boxtimes I'})$, $b' \in \mathbf{F}^{\boxtimes I'}$ *and* $b'' \in \mathbf{F}^{\boxtimes I''}$. *We have*

$$m^{(I')}\left(\mu^{\omega}_{I'}(\alpha_{I'})(e^{\mathcal{P}_{I'I''}}(b' \boxtimes b'' \cdot f))\right) = \\ e^{\mathcal{P}_{\{\bullet\}I''}} m^{(I')}\left(\mu^{\omega}_{I'}(\alpha_{I'})(b' \boxtimes b'' \cdot f)\right) \in \mathbf{J}^{\{\bullet\}\sqcup I''}_{\mathbb{A}^d}(\mathcal{B} \boxtimes \mathcal{B}^{\boxtimes I''}) \otimes_{k[\lambda_\bullet]} k[\lambda_{I'}] \tag{3.3}$$

for all $f \in \mathbf{J}^{(I'),\vec{I''}}_{\mathbb{A}^d}(\mathcal{O}^{\boxtimes I'}_{\mathbb{A}^d} \boxtimes (\omega^{\blacklozenge}_{\mathbb{A}^d})^{\boxtimes I''})$ *(recall the notation in the proof of Theorem 1.7).*

Proof. Since $e^{\mathcal{P}_{I'I''}}$ is a $\mathcal{D}$-module map, we have

$$m^{(I')}\left(\mu^{\omega}_{I'}(\alpha_{I'})(e^{\mathcal{P}_{I'I''}}(b' \boxtimes b'' \cdot f))\right) = m^{(I')}\left(e^{\mathcal{P}_{I'I''}}(\mu^{\omega}_{I'}(\alpha_{I'})(b' \boxtimes b'' \cdot f))\right).$$

Then the lemma follows from the fact that (recall the proof of Propostition 3.2)

$$m^{(I')}\left(e^{\mathcal{P}_{I'\sqcup I''}} -\right) = e^{\mathcal{P}_{\{\bullet\}\sqcup I''}} m^{(I')}\left(-\right).$$

□

We can now prove the following.

Proposition 3.7. *The chiral operations* $\{\mu_I^{\mathcal{F}}\}$ *define an* L_∞*-chiral algebra structure on* $\mathcal{F} = \mathbf{F} \otimes_{\mathcal{O}_{\mathbb{A}^d}} \omega^{\blacklozenge}_{\mathbb{A}^d}$ *for any graded vector space F.*

Proof. From the definition of $\{\mu_I^{\mathcal{F}}\}$ and the above lemma, we have

$$\begin{aligned}
&\mu_I^{\mathcal{F}}\left(\mathbf{d}(\alpha_{I'} b' \boxtimes b'' \cdot f)\right) \\
&= \mu_I^{\mathcal{F},comm}\left(e^{\mathcal{P}_I}\mathbf{d}(\alpha_{I'} b' \boxtimes b'' \cdot f)\right) = \mu_I^{\mathcal{F},comm}\left(\mathbf{d}(e^{\mathcal{P}_I}(\alpha_{I'} b' \boxtimes b'' \cdot f))\right) \\
&= \sum \mu^{\mathcal{F},comm}_{\bullet \sqcup I - I'} \circ \mu^{\mathcal{F},comm}_{I' \subset I}\left(e^{\mathcal{P}_{I'}} e^{\mathcal{P}_{I'I''}} e^{\mathcal{P}_{I''}}(\alpha_{I'} b' \boxtimes b'' \cdot f)\right) \\
&= \sum \mu^{\mathcal{F},comm}_{\bullet \sqcup I - I'} \circ \left(e^{\mathcal{P}_{\bullet I''}} e^{\mathcal{P}_{I''}} \mu^{\mathcal{F},comm}_{I' \subset I}(e^{\mathcal{P}_{I'}} \alpha_{I'} b' \boxtimes b'' \cdot f)\right) \\
&= \sum \mu^{\mathcal{F}}_{\bullet \sqcup I - I'} \circ \mu^{\mathcal{F}}_{I' \subset I}\left(\alpha_{I'} b' \boxtimes b'' \cdot f\right),
\end{aligned}$$

which proves the theorem. □

Remark 3.8. This is the Wick theorem in higher dimensions, which may be more familiar in the context of the operator product expansion for vertex algebras. See [10, 11] for the $d = 1$ Wick theorem in the language of chiral algebras.

Remark 3.9. Note that while we used the commutative chiral operations as part of the definition of $\{\mu_I^{\mathcal{F}}\}$, this new chiral algebra is *not* commutative.

From proposition 1.12 we know that the de Rham functor applied to $\mathcal{F}$ yields an L_∞ algebra $h(\mathcal{F})$. In this case, this L_∞ algebra is, in fact, a graded Lie algebra. Explicitly, this graded Lie algebra has underlying graded vector space

$$\mathrm{Sym}\left(F \otimes \omega^{-1}\right) \otimes \omega[d-1]. \tag{3.4}$$

The bracket is the natural extension of $\langle -, - \rangle$ by the graded Leibniz rule. In particular, $h(\mathcal{F})$ is a graded Poisson algebra.

3.3. **The Faonte-Hennion-Kapranov central extension via chiral operations.**

In this section, denote $\mathsf{A}_d^\bullet := \mathbf{J}_{\mathring{\mathbb{A}}^d}$ the Jouanolou model of punctured affine space $\mathring{\mathbb{A}}^d = \mathbb{A}^d \setminus \{0\}$. Equivalently, this is the quotient of the Jouanolou model of two points in $\mathbb{A}^d$ by the translations. It is a commutative dg algebra model for the derived global sections of the structure sheaf of the punctured affine space $\mathring{\mathbb{A}}^d$. For a Lie algebra $\mathfrak{g}$, introduce the following dg Lie algebra

$$\mathfrak{g}_d^\bullet = \mathfrak{g} \otimes_{\mathbf{k}} \mathsf{A}_d^\bullet.$$

This dg Lie algebra was introduced in [5] as a higher-dimensional generalization of current algebras which appear in symmetries of chiral conformal field theory [12]. One of the main results of [5] is that invariant polynimials of degree $(d+1)$ on the Lie algebra $\mathfrak{g}$ give rise to nontrivial L_∞ central extensions of this dg Lie algebra. The centrally extended algebras provide higher-dimensional, or multivariate, generalizations of the famous affine Kac–Moody algebras.

Tied to the usual affine Kac–Moody algebra is a well-known chiral (and vertex) algebra. In fact, it is a special case of a general enveloping algebra construction of a chiral algebra from any Lie* algebra. The notion of a Lie* algebra can be generalized to higher dimensions. While we do not purse the construction here, we do expect a universal enveloping-type construction which associates to such a higher-dimensional Lie* algebra a higher chiral algebra in the sense of this work. We leave this problem to future work.

Instead, we will give a more direct interpretation of the higher-dimemnsional Kac–Moody algebra in terms of its ”free field realization” using the higher Wick’s theorem of the previous section.

Let V be a $\mathfrak{g}$-representation and consider the graded $\mathcal{O}$-module

$$F = \underline{V^*} \oplus \underline{V} \otimes \omega[d-1] = V^* \otimes \mathcal{O} \oplus V \otimes \omega[d-1]$$

equipped with the obvious pairing in the sense of (3.1). (We will assume that V is an ordinary vector space, rather than a graded one, for simplicity.) The free ghost chiral algebra $\mathcal{F}$ associated to V, as defined in the previous section, can be identified, as graded $\mathcal{D}$-module, with

$$\mathcal{F} = \mathrm{Sym}(\underline{V^\vee}_{\mathcal{D}} \otimes \omega^{-1} \oplus \underline{V}[1-d] \otimes \omega_{\mathcal{D}} \otimes \omega^{-1}) \otimes \omega[d-1]. \tag{3.5}$$

The $\mathfrak{g}$-representation V defines the cochain map

$$\rho\colon \mathfrak{g}_d^\bullet \to (V^\vee \otimes_{\mathbf{k}} V) \otimes_{\mathbf{k}} \mathsf{A}_d^\bullet \subset h^{Lie}(\mathcal{F}) := (h \boxtimes 1)\left(\mathbf{J}^{2,\bullet}_{\mathbb{A}^d}(\mathcal{F} \boxtimes \mathcal{O})\right)$$

Proposition 3.10. *The map $\rho\colon \mathfrak{g}_d^\bullet \to h^{Lie}(\mathcal{F})$ is a map of dg Lie algebras.*

Proof. This follows from L_∞-relations. We have

$$\sum \mu_2 \circ \mu_2 \left(a_1 \boxtimes a_2 \boxtimes a_3 \cdot \alpha_{1*}\alpha_{2*}\alpha_{3*}\right) = \mu_3 \left(\mathbf{d}(a_1 \boxtimes a_2 \boxtimes a_3) \cdot \alpha_{1*}\alpha_{2*}\alpha_{3*}\right) = 0.$$

☐

Now, we move onto the central extension. Following [5], we consider the following graded skew-symmetric functional $\gamma_P : \left(\mathfrak{g}_d^\bullet[1]\right)^{\otimes_{\mathbf{k}} d+1} \to \mathbf{k}$

$$\gamma_P\left((x_0 \otimes f_0) \otimes (x_1 \otimes f_1) \otimes \cdots \otimes (x_d \otimes f_d)\right) = P(x_0, \ldots, x_d) \cdot \mathrm{Res}\left(f_0 \cdot \partial f_1 \wedge \cdots \wedge \partial f_d\right).$$

Here P is given by

$$P(x_0, \ldots, x_d) = \frac{1}{(d+1)!} \sum_{\sigma \in S_{n+1}} \mathrm{tr}_V \left(x_{\sigma(0)} \cdots x_{\sigma(d)}\right).$$

Where tr_V is the trace taken in the representation V.

Proposition 3.11. *Define the map*

$$\gamma^{ch} : \left(\mathbf{J}^{2,\bullet}_{\mathbb{A}^d}(\mathcal{F} \boxtimes \mathcal{O})[1]\right)^{\otimes_{\mathbf{k}} d+1} \to \mathbf{k}[z^r]^{1 \le r \le n}$$

by sending $(f(z_1, z)a_1 \boxtimes 1) \otimes \cdots \otimes (f(z_{d+1}, z)a_{d+1} \boxtimes 1)$ to

$$\mathrm{Res}\left(\mu_{d+1}(a_1 \boxtimes \cdots \boxtimes a_{d+1}) \cdot f_1(z_1, z) \cdots f_{d+1}(z_{d+1}, z)\right)$$

where μ_{d+1} is the $(d+1)$st chiral operation underlying $\mathcal{F}$ as defined in the previous section. Then the cocycle γ_P defined in [5] *is given by the composition $\gamma^{ch} \circ \iota$.*

Proof. The only Feynman graph which contributes non-trivially to the residue is the wheel graph with $(d+1)$-vertices, where there is a single Wick contraction between neighboring vertices. In [12] we have evaluated the weight of this diagram in line with the stated result. For another computation of the case when $d = 2$, see section 4.4. ☐

3.4. **Higher Virasoro structures.** Let $\mathfrak{witt}_d^\bullet$ denote the Jouanolou model of the derived global sections of the tangent sheaf on punctured affine space $\mathring{\mathbb{A}}^d$. When $d = 2$, a two-dimensional space of central extensions of $\mathfrak{witt}_2^\bullet$ is studied in [21]; the resulting algebras are higher-dimensional analogs of the ordinary Virasoro algebra. This two-dimensional space is naturally identified with span of the universal characteristic classes $c_1^3, c_1c_2 \in H^6(BU(2))$. Here is an explicit construction of the Virasoro central extensions when $d = 2$ using chiral operations and a similar free-field realization as above.

Recall that $\mathfrak{witt}_2^\bullet \simeq \mathbf{J}^\bullet_{\mathring{\mathbb{A}}^2} \cdot \partial_{z^1} \oplus \mathbf{J}^\bullet_{\mathring{\mathbb{A}}^2} \cdot \partial_{z^2}$. We take $V = \mathbb{C} \cdot \mathbf{v}$ to be a one-dimensional vector space and denote $\mathcal{F}$ to be the corresponding free ghost chiral algebra defined by (3.5). We construct the following map

$$\rho_{vir}\colon \mathfrak{witt}_2^\bullet \to h^{Lie}(\mathcal{F}) := (h \boxtimes 1)\left(\mathbf{J}^{2,\bullet}_{\mathring{\mathbb{A}}^d}(\mathcal{F} \boxtimes \mathcal{O})\right)$$

defined by

$$\rho_{vir}\Big(\sum_{s=1}^{2} T^s(z)\partial_{z^s}\Big) = \sum_{s=1}^{2} T^s(z-w)\cdot(\mathbf{v}^\vee \otimes \partial_{z^s})\cdot \mathbf{v} \boxtimes 1, \quad T^s(z) \in \mathbf{J}^\bullet_{\mathring{\mathbb{A}}^2}.$$

Parallel to the previous section, we define

$$\gamma_{vir}\colon \wedge^3 \mathfrak{witt}_2^\bullet \to \mathbb{C}$$

by the formula

$$\gamma_{vir}(T_1, T_2, T_3) := \operatorname{Res}\left(\mu_3^{\mathcal{F}}(\rho_{vir}(T_1) \boxtimes \rho_{vir}(T_2) \boxtimes \rho_{vir}(T_3))\right).$$

Using the explicit result in the next section, γ_{vir} can be computed explicitly.

Proposition 3.12. *γ_{vir} is a cocycle and*

$$\gamma_{vir}(T_1, T_2, T_3) = \frac{1}{72}\operatorname{Res}(\operatorname{div}(T_1)\cdot\langle\partial T_1, \partial T_2\rangle + \operatorname{div}(T_2)\cdot\langle\partial T_1, \partial T_2\rangle + \operatorname{div}(T_3)\cdot\langle\partial T_1, \partial T_2\rangle)$$

$$-\frac{1}{72}\operatorname{Res}(\mathrm{div}(\mathrm{T}_1)\cdot\partial\mathrm{div}(\mathrm{T}_2)\cdot\partial\mathrm{div}(\mathrm{T}_3)).$$

Here $\operatorname{div}(T) = \sum_{s=1}^{2} \partial_{z^s} T^s(z)$ *is the divergence and*

$$\langle T, R\rangle := \sum_{r=1}^{2}\sum_{s=1}^{2} \partial_{z^r} T^s(z) \cdot \partial_{z^s} R^r(z).$$

In the above, we extend $\langle -, -\rangle$ to one form valued vector field $\partial T := \sum_{s=1}^{2} \partial T^s(z)\partial_{z^s}$.

Proof. Direct computation by using the result in section 4.4. □

Remark 3.13. This result is compatible with a local universal index theorem in the spirit of [3]. Indeed, up to an overall factor, this central extension corresponds to the degree six part of the universal Todd class $\mathrm{Td}_6 = \frac{1}{24}c_1c_2 = \frac{1}{24}(c_1^3 - c_1\,\mathrm{ch}_2)$, where ch_2 is the second component of the universal Chern character.

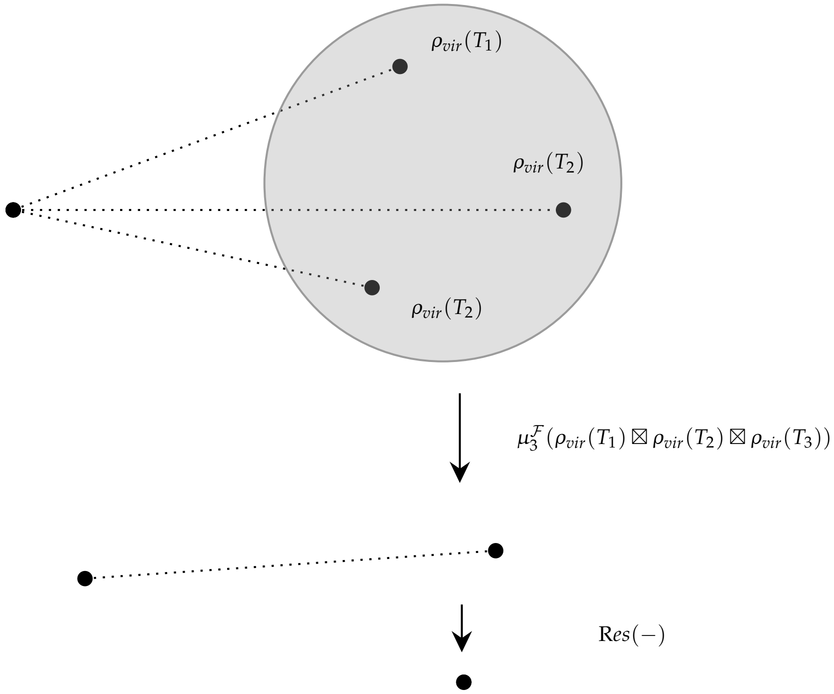

$\rho_{vir}(T_1)$
$\rho_{vir}(T_2)$
$\rho_{vir}(T_2)$
$\mu_3^{\mathcal{F}}(\rho_{vir}(T_1) \boxtimes \rho_{vir}(T_2) \boxtimes \rho_{vir}(T_3))$
$\mathrm{R}es(-)$

FIGURE 1. Virasoro central extension via the chiral operation

Remark 3.14. More generally, in dimension d, we expect an isomorphism $H^{2d+2}(BU(d)) \cong H^2_{\mathrm{Lie}}(\mathfrak{witt}_d)$. Thus, we conjecture a complete characterization of central extensions of the d-dimensional Witt algebra in terms of universal characteristic classes.

4. THE UNIT CHIRAL ALGEBRA IN $\mathbb{A}^2$ IN DETAIL

In this section we unpack the two-dimensional chiral operations in greater detail. For simplicity, we denote by μ_k the k-ary chiral operation for the unit chiral algebra $\omega_{\mathbb{A}^2}[1]$ as defined in section 2.4. We will assume that $\mathbf{k} = \mathbb{C}$ in this section. However, the main theorem (Theorem 4.13) works for the general ground field $\mathbf{k}$ if one accepts the existence of a unit chiral algebra over $\mathbf{k}$.

4.1. **The propagator.** First, we begin with the definition.

Definition 4.1. The propagator P_{12} is an element of $\mathbf{J}_{\mathbb{A}^2}^{\{1,2\}}$, which is given by

$$P_{12} = x_{12}^1 \mathbf{d} x_{12}^2 - x_{12}^2 \mathbf{d} x_{12}^1.$$

From now on, we will use the notation

$$(f|g) = f^1 g^1 + f^2 g^2, \quad f \wedge g = f^1 g^2 - f^2 g^1$$

for variables f, g with superscript $1, 2$. For variables $(\lambda^1, \lambda^2), (x^1, x^2), (z^1, z^2)$, we define the antisymmetric symbols

$$\lambda \wedge z = \lambda^1 z^2 - \lambda^2 z^1 = z^2 \lambda^1 - z^1 \lambda^2 = -z \wedge \lambda$$

and

$$\lambda \wedge x = \lambda^1 x^2 - \lambda^2 x^1 = x^2 \lambda^1 + x^1 x^2 - x^1 \lambda^2 - x^1 x^2 = -x \wedge \lambda.$$

We will denote by $d\mathbf{z}_i^{\blacklozenge}, i \in I$ the shifted top form

$$d\mathbf{z}_i^{\blacklozenge} = dz_i^1 \wedge dz_i^2[1], \quad d\mathbf{z}_I^{\blacklozenge} := \boxtimes_{i \in I} d\mathbf{z}_i^{\blacklozenge}.$$

Our first result is the simplest nontrivial computation of the chiral two-ary operation.

Proposition 4.2. *We have the following identity:*

$$\mu_2(P_{12} d\mathbf{z}_1^{\blacklozenge} \boxtimes d\mathbf{z}_2^{\blacklozenge}) = d\mathbf{z}^{\blacklozenge}.$$

Proof. Note P_{12} contains two terms. We first find the Feynman graphs which correspond to these two monomials. Let $\vec{\Gamma}$ be the directed graph constructed as follows:

- $\vec{\Gamma}_0 = \{1, 2\}$ and $\vec{\Gamma}_1 = \{e_1, e_2\}$.
- $t(e_1) = t(e_2) = 1$ and $h(e_1) = h(e_2) = 2$.

We consider two Feynman graphs $(\vec{\Gamma}, m_1, 2)$ and $(\vec{\Gamma}, m_2, 2)$, where m_1 and m_2 is defined by

$$\begin{cases} m_1(e_1) = 1, m_1(e_2) = 2. \\ m_2(e_1) = 2, m_2(e_2) = 1. \end{cases}$$

Then we have $p_{(\vec{\Gamma}, m_1, 2)} = x_{12}^1 \mathbf{d} x_{12}^2$ and $p_{(\vec{\Gamma}, m_2, 2)} = x_{12}^2 \mathbf{d} x_{12}^1$. The graphical Green's function is:

$$\begin{cases} d_{\vec{\Gamma}}^{-1}(t)_{e_1 1} = \frac{t_{e_2}}{t_{e_1} + t_{e_2}}, \\ d_{\vec{\Gamma}}^{-1}(t)_{e_2 1} = \frac{t_{e_1}}{t_{e_1} + t_{e_2}}. \end{cases}$$

By proposition 2.15, we have

$$
\begin{aligned}
&\mu_2(p_{(\vec{\Gamma},m_1,2)}d\mathbf{z}_1^{\blacklozenge}\boxtimes d\mathbf{z}_2^{\blacklozenge})\\
&=\Big(\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})}\iota_{d^2\tilde{z}_1 d^2\bar{\tilde{z}}_1}\\
&\circ(-d\hat{y}_{e_1}^1)\circ(-(\sum_{s=1}^{2}(\partial_{\lambda_{e_2}^s}+\tilde{z}_{e_2}^s)\circ d\hat{y}_{e_2}^s)\circ d\hat{y}_{e_2}^2)(d^2z_1\boxtimes d^2z_2)\Big)\Big|_{\tilde{z}_1=0,\tilde{z}_2=w}\\
&=\Big(\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})}\frac{t_{e_1}}{t_{e_1}+t_{e_2}}\Big(\frac{t_{e_1}}{t_{e_1}+t_{e_2}}d(\frac{t_{e_2}}{t_{e_1}+t_{e_2}})-\frac{t_{e_2}}{t_{e_1}+t_{e_2}}d(\frac{t_{e_1}}{t_{e_1}+t_{e_2}})\Big)\Big)\,d^2w.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\mu_2(-p_{(\vec{\Gamma},m_2,2)}d\mathbf{z}_1^{\blacklozenge}\boxtimes d\mathbf{z}_2^{\blacklozenge})\\
&=\Big(\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})}\frac{t_{e_2}}{t_{e_1}+t_{e_2}}\Big(\frac{t_{e_1}}{t_{e_1}+t_{e_2}}d(\frac{t_{e_2}}{t_{e_1}+t_{e_2}})-\frac{t_{e_2}}{t_{e_1}+t_{e_2}}d(\frac{t_{e_1}}{t_{e_1}+t_{e_2}})\Big)\Big)\,d^2w.
\end{aligned}
$$

Combining these computations, and using proposition 2.17, we can get

$$
\begin{aligned}
&\mu_2(P_{12}d\mathbf{z}_1^{\blacklozenge}\boxtimes d\mathbf{z}_2^{\blacklozenge})\\
&=\Big(\int_{S^+((0,+\infty)^{\vec{\Gamma}_1})}\Big(\frac{t_{e_1}}{t_{e_1}+t_{e_2}}d(\frac{t_{e_2}}{t_{e_1}+t_{e_2}})-\frac{t_{e_2}}{t_{e_1}+t_{e_2}}d(\frac{t_{e_1}}{t_{e_1}+t_{e_2}})\Big)\Big)\,d^2w\\
&=\Big(\int_{\{(t_{e_1},t_{e_2})\in(0,+\infty)^2|t_{e_1}+t_{e_2}=1\}}\Big(\frac{t_{e_1}}{t_{e_1}+t_{e_2}}d(\frac{t_{e_2}}{t_{e_1}+t_{e_2}})-\frac{t_{e_2}}{t_{e_1}+t_{e_2}}d(\frac{t_{e_1}}{t_{e_1}+t_{e_2}})\Big)\Big)\,d^2w
\end{aligned}
$$

The integral above produces 1, as desired. ☐

Remark 4.3. This element is closed. In fact, any closed 2-operation is a multiple of μ_2.

4.2. **Residues as two-ary chiral operations.** Our next result is to compute the value of the two-ary chiral operation on arbitrary derivatives of the propagator. To do this, we will introduce a generating function expression for derivatives of the propagator P_{12}. Instead an explicit Feynman diagram calculation, we will characterize the expression by using the D-module relations (together with the computation from the previous section).

Introduce new formal variables $\mathfrak{z}_{12}=(\mathfrak{z}_{12}^1,\mathfrak{z}_{12}^2)$ and form the generating series

$$
P_{12}(\mathfrak{z}_{12}):=\sum_{r,s\geq 0}\frac{1}{r!s!}(\partial_{z_1^1})^r(\partial_{z_1^s})^s P_{12}(\mathfrak{z}_{12}^1)^r(\mathfrak{z}_{12}^2)^s.
$$

Lemma 4.4. *We have*

$$
P_{12}(\mathfrak{z}_{12})=\frac{P_{12}}{(1+(x_{12}|\mathfrak{z}_{12}))^2},\quad x_{12}^s(\mathfrak{z}_{12})=\frac{x_{12}^s}{1+(x_{12}|\mathfrak{z}_{12})},\quad s=1,2.
$$

The above equalities are understood as power series expansion of $\mathfrak{z}_{12}$.

Proof. From proposition 1.2, we have

$$(-x_{12}^{i_1})\cdots(-x_{12}^{i_k})\cdot P_{12}=\frac{\partial_{z_1^{i_1}}\cdots\partial_{z_1^{i_k}}P_{12}}{(k+1)!},\quad (-x_{12}^{i_1})\cdots(-x_{12}^{i_k})\cdot x_{12}^{s}=\frac{\partial_{z_1^{i_1}}\cdots\partial_{z_1^{i_k}}x_{12}^{s}}{k!},\quad s=1,2$$

and the lemma follows. □

Proposition 4.5. *We have the following explicit formula for* μ_2

$$\mu_2\left(P_{12}(\mathfrak{z}_{12})\cdot d\mathbf{z}_1^{\blacklozenge}\boxtimes d\mathbf{z}_2^{\blacklozenge}\right)=d\mathbf{z}^{\blacklozenge}\otimes e^{-(\lambda_1|\mathfrak{z}_{12})}.$$

Proof. We only need to determine the value of

$$\mu_2\left((x_{12}^1)^{m_1}\cdot(x_{12}^2)^{m_2}\mathbf{d}x_{12}^1\cdot d\mathbf{z}_1^{\blacklozenge}\boxtimes d\mathbf{z}_2^{\blacklozenge}\right),\quad m_1,m_2\geq 1$$

Note that

$$\begin{aligned}&\partial_{z_1^1}\left((x_{12}^1)^{m_1-1}\cdot(x_{12}^2)^{m_2}\mathbf{d}x_{12}^1\right)\\&=(-(m_1-1)-m_2-2)\cdot(x_{12}^1)^{m_1}\cdot(x_{12}^2)^{m_2}\mathbf{d}x_{12}^1.\end{aligned}$$

We find

$$(x_{12}^1)^{m_1}\cdot(x_{12}^2)^{m_2}\mathbf{d}x_{12}^1=\frac{-1}{m_1+m_2+1}\cdots\frac{-1}{m_2+1}\partial_1^m\left((x_{12}^2)^{m_2}\mathbf{d}x_{12}^1\right).$$

If $m_2>1$, we can further reduce $\mu_2\left((x_{12}^2)^{m_2}\mathbf{d}x_{12}^1\right)$ to $-m_2\cdot\mu_2\left(x_{12}^1\cdot(x_{12}^2)^{m_2-1}\mathbf{d}x_{12}^2\right)$. Repeat the same calculation, we have

$$\begin{aligned}&\partial_{z_1^2}\left(x_{12}^1\cdot(x_{12}^2)^{m_2-2}\mathbf{d}x_{12}^2\right)\\&=(-1-m_2+2-2)\cdot\left(x_{12}^1\cdot(x_{12}^2)^{m_2-1}\mathbf{d}x_{12}^2\right).\end{aligned}$$

Thus

$$\left(x_{12}^1\cdot(x_{12}^2)^{m_2-1}\mathbf{d}x_{12}^2\right)=\frac{-1}{m_2+1}\cdots\frac{-1}{3}\partial_{z_1^2}^{m_2-1}\left(x_{12}^1\cdot\mathbf{d}x_{12}^2\right).$$

□

Proposition 4.6 (Arnold relation)**.**

$$P_{12}P_{13}+P_{12}P_{23}+P_{13}P_{23}=\mathbf{d}\left(P_{13}\cdot x_{12}\wedge x_{23}+P_{12}\cdot x_{13}\wedge x_{23}+P_{23}\cdot x_{12}\wedge x_{13}\right)$$

Proof.

$$\begin{aligned}P_{12}P_{13}&=P_{12}P_{13}(x_{23}^1z_{23}^1+x_{23}^2z_{23}^2)\\&=P_{12}P_{13}\left(x_{23}^1(z_{13}^1-z_{12}^1)+x_{23}^2(z_{13}^1-z_{12}^1)\right)\\&=P_{13}\cdot dx_{12}\wedge x_{23}+P_{12}\cdot dx_{13}\wedge x_{23}.\end{aligned}$$

□

Corollary 4.7.

$$P_{12}P_{13}P_{23} = \mathbf{d}\left(P_{23}P_{13} \cdot x_{12} \wedge x_{23} + P_{23}P_{12} \cdot x_{13} \wedge x_{23}\right).$$

Remark 4.8. The (derived) Arnold relation holds for general $\mathbb{C}^n, n \geq 2$ as well. This is proved in [6] for the polysimplicial model. To compare, one can replace x^l_{ij} with $\frac{u^l_{ij}}{z^l_{ij}}$ to obtain the Arnold relation in the Jouanolou model.

4.3. **Formulas for chiral operations.** In section 2, we constructed a series of chiral operations using Feynman integrals and show that they satisfy a L_∞-relation–this was precisely the L_∞ relations underlying the unit chiral algebra on $\mathbb{A}^d$. In the case of $\mathbb{A}^2$, the L_∞-relation puts a very strong constraint on the chiral operations and sometimes can sometimes completely fix them. In this section, we derive recursive formulas for chiral operations purely from the L_∞-relation and find a perfect match with numerous calculations that have appeared in the work of physics [4].

4.3.1. *Recursive formula for Type 1 Laman graphs.* For the standard graph terminology used in this section we refer the reader to [9]. Here we change our convention of graphs used in section 2, now a graph Γ is simply a pair $\Gamma = (V(\Gamma), E(\Gamma))$ of vertices and edges (we do not allow multiple edges between two vertices anymore).

Definition 4.9 (Laman)**.** Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph with vertices $V(\Gamma)$ and edges $E(\Gamma)$. We say Γ is a *Laman graph* if $|E(\Gamma)| = 2|V(\Gamma)| - 3$ and $|E(\Gamma')| \leq 2|V(\Gamma')| - 3$ for every subgraph Γ' of Γ.

Remark 4.10. When $d = 2$, the concept of Laman graph originate from Laman's characterization of generic rigidity of graphs embeded in two dimensional real linear space (see [15]). Our definition appears in [4].

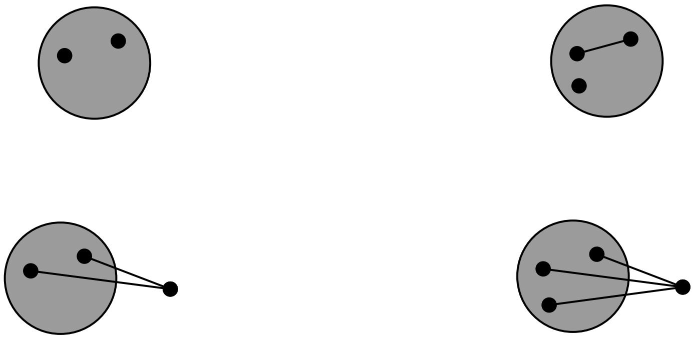
Starting with a single edge, a Laman graph can be constructed by a sequence of the following two types of Henneberg operations (see Fig. 2):

- Henneberg I: the new vertex is connected via two new edges to two old vertices.
- Henneberg II: remove an edge and add a new vertex connected to its two endpoints and to some other vertex.

In this section we consider the following variant of Henneberg I.

Definition 4.11 (Henneberg I′)**.** We define the following type operation:

- Henneberg I′: the new vertex is connected via two new edges to two old vertices which are end points of an old edge.



Henneberg I

Henneberg II

FIGURE 2. Two types of Henneberg operations

See Fig. 3.

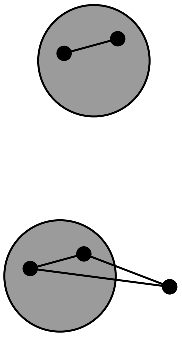


FIGURE 3. Henneberg I′ operation

We will mainly focus on a special class of Laman graphs which we call Type I′.

Definition 4.12 (Type I′ Laman graphs)**.** A Laman graph is called *Type I′* if it can be constructed by a sequence of Henneberg I′ operations.

Suppose Γ is a Type I′ Laman graph. We fix a vertex $o \in V(\Gamma)$ and an edge $e = [ov] \in E(\Gamma)$. Then, Γ can be constructed by a following sequence of Hennerberg operations

$$\mathbf{H}_\Gamma(o,e) = [(e_0 = e, v_1), (e_1, v_2), \dots, (e_{k-1}, v_k)].$$

Here, the vertex set of the graph is $V(\Gamma) = \{o, v, v_1, \dots, v_k\}$. From the edge $e_0 = e = [ov]$ we add a new vertex v_1 and draw two edges connecting v_1 and o, v. Repeating, at the ith stage we add a new vertex v_i and choose an edge e_{i-1} of the graph (we allow

$e_{i-1} = e_{i'}, i' < i-1$) and then connect v_i and the vertices of e_{i-1} by two edges. This can be illustrated by the picture as follows.

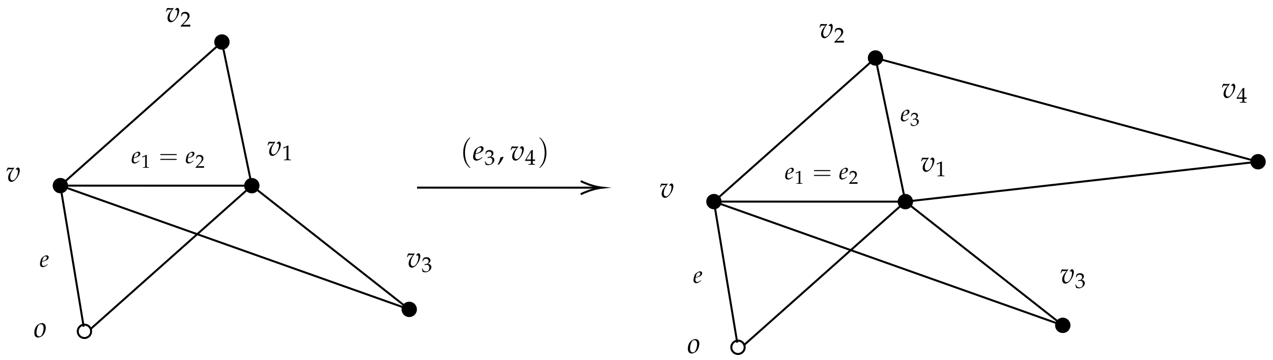


FIGURE 4.

A Type I′ Laman graph can be constructed from different sequences of Hennerberg moves $\mathbf{H}_\Gamma(o,e), \mathbf{H}'_\Gamma(o,e)$ starting from o, e. Using the Arnold relation, we can show W_Γ is exact in the Jouanolou model and thus by the L_∞-relation for chiral operations we reduce to the chiral operation for small graphs. The Type I′ condition ensure that the smaller graphs are still Type I′ Laman. In this section, we will give the detailed construction of this outline thus providing an explicit recursive formula for the chiral operation on Type I′ Laman graphs.

Theorem 4.13. *Given a Type I′ Laman graph* Γ *and a sequence of Henneberg I′ operations* $\mathbf{H}_\Gamma(o,e)$ *which constructs* Γ*, there are polynomial functions*

$$\{f_{v,\vec{e}}^{\Gamma,\mathbf{H}_\Gamma(o,e)}\}_{v\in V(\Gamma)-\{o\}} \in \mathbb{C}[r_k, s_k]_{k=1,\dots,|V(\Gamma)|-2}, \quad \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)} \in \mathbb{C}[r_k, s_k, \lambda_i^l]^{i\in V(\Gamma)-\{o\}, l=1,2}_{k=1,\dots,|V(\Gamma)|-2},$$

such that

$$\mu_{V(\Gamma)}\left(W_\Gamma(\mathfrak{z}_{\vec{e}})\right) = \int_{([0,1]\times[0,1])^{|\mathbf{H}_\Gamma(o,e)|}} e^{\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}} \cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)} \cdot \bigwedge_{k=1,\dots,|\mathbf{H}_\Gamma(o,e)|} (dr_k \wedge ds_k)$$

where

$$\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)} = \mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)}(\lambda;\mathfrak{z}_{\vec{e}}) := \sum_{v\in V(\Gamma)-\{o\}} (\lambda_v | \sum_{e\in E(\Gamma)} f_{v,\vec{e}}^{\Gamma,\mathbf{H}_\Gamma(o,e)} \mathfrak{z}_{\vec{e}}).$$

We can construct a new graph $\Gamma^\star$ *using a Henneberg I′ operations* $\mathbf{H}_{\Gamma^\star}(o,e) = [\mathbf{H}_\Gamma(o,e), (e^\triangleright, \star)]$*. Then we have the following recursive formulas for* $\mu_{V(\Gamma^\star)}\left(W_{\Gamma^\star}(\mathfrak{z}_{\vec{e}})\right)$*:*

1) If $\star$ *is connected to* i, j *(i.e.,* $\vec{e^{\triangleright}} = ij$*), then we introduce new variables*

$$\begin{cases} \mathfrak{z}^{\triangleright}_{ij} = \mathfrak{z}_{ij} + (1-r_\star)(\mathfrak{z}_{\star j} - \mathfrak{z}_{\star i} - \mathfrak{z}_{ij}), & \\ \mathfrak{z}^{\triangleright}_{\vec{e}} = \mathfrak{z}_{\vec{e}}, & \text{if } \vec{e} \neq ij, \\ \lambda^{\triangleright}_i = \lambda_i + (1-s_\star)\lambda_\star, & \\ \lambda^{\triangleright}_j = \lambda_j + s_\star\lambda_\star, & \\ \lambda^{\triangleright}_v = \lambda_v, & \text{if } v \neq i, j. \end{cases}$$

$$\begin{aligned} \mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} &= \mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)}(\lambda^{\triangleright};\mathfrak{z}^{\triangleright}_{\vec{e}}) - \left(\lambda_\star|(1-s_\star)\mathfrak{z}_{\star i} + s_\star\mathfrak{z}_{\star j}\right) \\ &= \sum_{v\in V(\Gamma)-o} (\lambda^{\triangleright}_v| \sum_{e\in E(\Gamma)} f^{\mathbf{H}_\Gamma}_{v,\vec{e}}\mathfrak{z}^{\triangleright}_{\vec{e}}) - \left(\lambda_\star|(1-s_\star)\mathfrak{z}_{\star i} + s_\star\mathfrak{z}_{\star j}\right). \end{aligned}$$

$$\begin{aligned} \mathbf{G}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} &= \left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda;\mathfrak{z}_{\vec{e}}) \wedge \lambda_\star\right)\cdot r_\star \cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright}) \\ &= \left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda;\mathfrak{z}_{\vec{e}}) \wedge \lambda_\star\right)\cdot r_\star \cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda_1,\dots,\lambda_i + (1-s_\star)\lambda_\star,\dots,\lambda_j + s_\star\lambda_\star,\dots,\lambda_n). \end{aligned}$$

1)If $\star$ *is connected to* i, o *(i.e.,* $\vec{e^{\triangleright}} = io$*), then we introduce new variables*

$$\begin{cases} \mathfrak{z}^{\triangleright}_{io} = \mathfrak{z}_{io} + (1-r_\star)(\mathfrak{z}_{\star o} - \mathfrak{z}_{\star i} - \mathfrak{z}_{io}), & \\ \mathfrak{z}^{\triangleright}_{\vec{e}} = \mathfrak{z}_{\vec{e}}, & \text{if } \vec{e} \neq io, \\ \lambda^{\triangleright}_i = \lambda_i + (1-s_\star)\lambda_\star, & \\ \lambda^{\triangleright}_v = \lambda_v, & \text{if } v \neq i. \end{cases}$$

$$\begin{aligned} \mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} &= \mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)}(\lambda^{\triangleright};\mathfrak{z}^{\triangleright}_{\vec{e}}) - (\lambda_\star|(1-s_\star)\mathfrak{z}_{\star i} + s_\star\mathfrak{z}_{\star o}) \\ &= \sum_{v\in V(\Gamma)-o} (\lambda^{\triangleright}_v| \sum_{e\in E(\Gamma)} f^{\mathbf{H}_\Gamma}_{v,\vec{e}}\mathfrak{z}^{\triangleright}_{\vec{e}}) - (\lambda_\star|(1-s_\star)\mathfrak{z}_{\star i} + s_\star\mathfrak{z}_{\star o}). \end{aligned}$$

$$\begin{aligned} \mathbf{G}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} &= \left(\partial_{\mathfrak{z}_{io}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda;\mathfrak{z}_{\vec{e}}) \wedge \lambda_\star\right)\cdot r_\star \cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright}) \\ &= \left(\partial_{\mathfrak{z}_{io}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda;\mathfrak{z}_{\vec{e}}) \wedge \lambda_\star\right)\cdot r_\star \cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda_1,\dots,\lambda_i + (1-s_\star)\lambda_\star,\dots,\lambda_n). \end{aligned}$$

Remark 4.14. The parameters r and s here are closely related to the Schwinger parameters in the previous section. However, the precise relation is complicated, which we left to future investigation.

Using the above formula, we can write the formulas for some loop diagrams and match with results in [4]. The remarkable thing is that, the chiral operation on Type I′ Laman graph is completely constrained by the L_∞-relations and can be solved recursively without knowing Feynman integrals. In other words, the recursive formula gives another way to compute the complicated Feynman integrals. We provide some explicit examples.

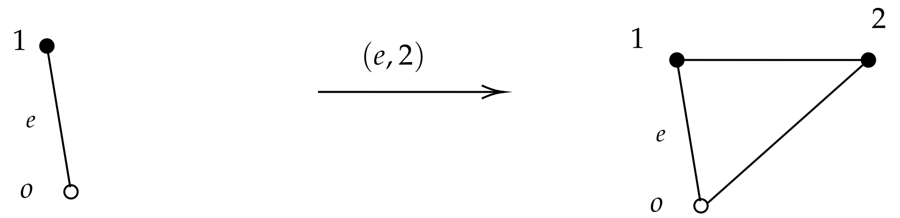


FIGURE 5.

4.4. **Example: one-loop diagrams.** We start from

$$\mathbf{W}_{\bullet\!-\!\bullet}(\lambda;\mathfrak{z}_{\vec{e}}) = -(\lambda_1|\mathfrak{z}_{1o}).$$

By the recursive formula, we have

$$\begin{aligned}
\mathbf{W}_{\triangle}(\lambda;\mathfrak{z}_{\vec{e}}) &= -(\lambda_1^{\triangleright}|\mathfrak{z}_{1o}^{\triangleright}) - (1-s_2)\,(\lambda_2|\mathfrak{z}_{21}) - s_2\,(\lambda_2|\mathfrak{z}_{2o}) \\
&= -\,(\lambda_1 + (1-s_2)\lambda_2|\mathfrak{z}_{1o} + (1-r_2)(\mathfrak{z}_{2o} - \mathfrak{z}_{21} - \mathfrak{z}_{1o})) - (1-s_2)\,(\lambda_2|\mathfrak{z}_{21}) - s_2\,(\lambda_2|\mathfrak{z}_{2o}) \\
&= -\,(\lambda_1|\mathfrak{z}_{1o}) - (\lambda_1|(1-r_2)(\mathfrak{z}_{2o} - \mathfrak{z}_{21} - \mathfrak{z}_{1o})) - ((1-s_2)\lambda_2|(1-r_2)(\mathfrak{z}_{2o} - \mathfrak{z}_{21} - \mathfrak{z}_{1o})) \\
&- ((1-s_2)\lambda_2|\mathfrak{z}_{1o}) - (1-s_2)\,(\lambda_2|\mathfrak{z}_{21}) - s_2\,(\lambda_2|\mathfrak{z}_{2o}) \\
&= -(\lambda_1|\mathfrak{z}_{1o}) - (\lambda_2|\mathfrak{z}_{2o}) - (1-r_2)\,(\lambda_1|\mathfrak{z}_{2o} - \mathfrak{z}_{21} - \mathfrak{z}_{1o}) + r_2(1-s_2)\,(\lambda_2|\mathfrak{z}_{2o} - \mathfrak{z}_{21} - \mathfrak{z}_{1o})\,,
\end{aligned}$$

$$\mathbf{G}_{\triangle} = r_2 \cdot (\partial_{\mathfrak{z}_{1o}} \mathbf{W}_{\bullet\!-\!\bullet}) \wedge \lambda_2 = r_2 \cdot \lambda_1 \wedge \lambda_2.$$

The 3-operation is given by the formula

$$\begin{aligned}
&\mu_{\{o,1,2\}}\left(P_{1o}(\mathfrak{z}_{1o})P_{12}(\mathfrak{z}_{12})P_{2o}(\mathfrak{z}_{2o})d\mathbf{z}^{\blacklozenge}_{\{o,1,2\}}\right) \\
&= \int_{[0,1]\times[0,1]} e^{\mathbf{W}_{\triangle}(\lambda;\mathfrak{z}_{\vec{e}})} \cdot \mathbf{G}_{\triangle} \cdot dr_2 ds_2 \cdot d\mathbf{z}_o^{\blacklozenge} \\
&= -e^{-(\lambda_2|\mathfrak{z}_{2o})-(\lambda_1|\mathfrak{z}_{1o})} \cdot \lambda_1 \wedge \lambda_2 \left(\frac{1}{(\lambda_1|\mathfrak{z})(\lambda_2|\mathfrak{z})} - \frac{e^{-(\lambda_1|\mathfrak{z})}}{(\lambda_1|\mathfrak{z})((\lambda_1+\lambda_2)|\mathfrak{z})} - \frac{e^{(\lambda_2|\mathfrak{z})}}{(\lambda_2|\mathfrak{z})((\lambda_1+\lambda_2)|\mathfrak{z})} \right) \cdot d\mathbf{z}_o^{\blacklozenge} \\
&= -e^{-(\lambda_2|\mathfrak{z}_{2o})-(\lambda_1|\mathfrak{z}_{1o})} \cdot \lambda_1 \wedge \lambda_2 \cdot \int_{l_1+l_2\le 1} e^{-l_1(\lambda_1|\mathfrak{z})+l_2(\lambda_2|\mathfrak{z})} dl_1 dl_2 \cdot d\mathbf{z}_o^{\blacklozenge},
\end{aligned}$$

where $\mathfrak{z} = \mathfrak{z}_{2o} - \mathfrak{z}_{21} - \mathfrak{z}_{1o}$.

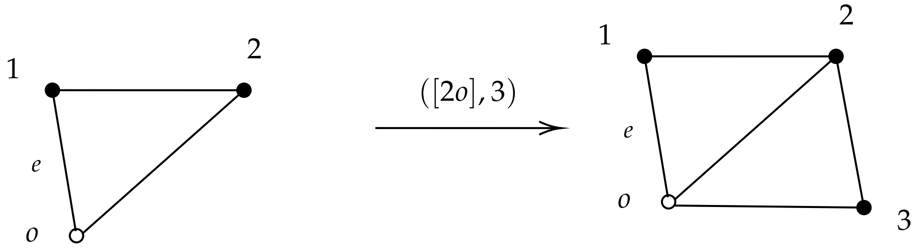


FIGURE 6.

4.5. Example: two-loop diagrams. Now, we start from the expression

$$\begin{aligned}\mathbf{W}_{\Diamond} &= \mathbf{W}_{\triangle}(\lambda^{\triangleright}; \mathfrak{z}_{\vec{e}}^{\triangleright}) - (\lambda_3 | (1-s_3)\mathfrak{z}_{32} + s_3\mathfrak{z}_{3o}) \\ &= -(\lambda_1|\mathfrak{z}_{1o}) - (\lambda_2|\mathfrak{z}_{2o}) - (\lambda_3|\mathfrak{z}_{3o}) + \Big(\lambda_1 | -(1-r_2)\mathfrak{z}_{[21o]} - (1-r_3)(1-r_2)\mathfrak{z}_{[32o]}\Big) \\ &+ \Big(\lambda_2 | (1-r_3)(r_2(1-s_2)-1)\mathfrak{z}_{[32o]} + r_2(1-s_2)\mathfrak{z}_{[21o]}\Big) \\ &+ \Big(\lambda_3 | r_2(1-s_2)(1-s_3)\mathfrak{z}_{[21o]} + (1-s_3)(1-(1-r_3)(1-r_2(1-s_2)))\mathfrak{z}_{[32o]}\Big)\end{aligned}$$

Here, we have defined the loop momentum $\mathfrak{z}_{[21o]} = \mathfrak{z}_{21} - \mathfrak{z}_{2o} - \mathfrak{z}_{1o}$, for example.

$$\begin{aligned}\mathbf{G}_{\Diamond} &= (\partial_{\mathfrak{z}_{2o}} \mathbf{W}_{\triangle}) \wedge \lambda_3 \cdot r_3 \cdot \mathbf{G}_{\triangle}(\lambda^{\triangleright}) \\ &= -\left(((1-r_2)\lambda_1 + (1-r_2(1-s_2))\lambda_2) \wedge \lambda_3\right) \cdot r_3 r_2 \cdot (\lambda_1 \wedge (\lambda_2 + (1-s_3)\lambda_3))\end{aligned}$$

We have the following result

$$\begin{aligned}&\mu_{\{o,1,2,3\}}\Big(P_{3o}(\mathfrak{z}_{3o})P_{32}(\mathfrak{z}_{32})P_{1o}(\mathfrak{z}_{1o})P_{12}(\mathfrak{z}_{12})P_{2o}(\mathfrak{z}_{2o})d\mathbf{z}^{\blacklozenge}_{\{o,1,2,3\}}\Big) \\ &= \int_{([0,1]\times[0,1])^2} e^{\mathbf{W}_{\Diamond}} \cdot \mathbf{G}_{\Diamond} \cdot dr_2 ds_2 dr_3 ds_3 \cdot d\mathbf{z}^{\blacklozenge}_o.\end{aligned}$$

In particular, the constant term (when $\mathfrak{z}_{\vec{e}} = 0$) is

$$\begin{aligned}&\mu_{\{o,1,2,3\}}\Big(P_{3o}P_{32}P_{1o}P_{12}P_{2o}d\mathbf{z}^{\blacklozenge}_{\{o,1,2,3\}}\Big) \\ &= \int_{([0,1]\times[0,1])^2} \mathbf{G}_{\Diamond} \cdot dr_2 ds_2 dr_3 ds_3 \cdot d\mathbf{z}^{\blacklozenge}_o \\ &= \frac{1}{24}(\lambda_1 \wedge (\lambda_3 + 2\lambda_2)) \cdot (\lambda_3 \wedge (\lambda_1 + 2\lambda_2)) \cdot d\mathbf{z}^{\blacklozenge}_o.\end{aligned}$$

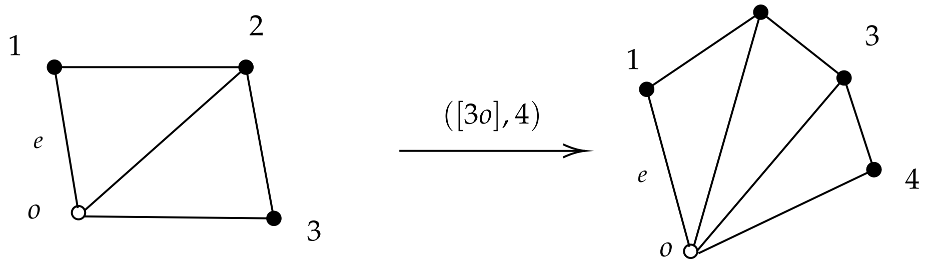
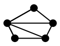
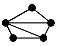
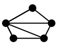


FIGURE 7.

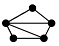
4.6. **Example: 3-loop diagrams.** By the same but tedious computation, one can obtain the formula for **W**. Here we only compute **G**. We have

G $= r_4 \cdot$ **G** $(\lambda^{\triangleright}) \cdot (\partial_{\mathfrak{z}_{3o}}$ **W** $) \wedge \lambda_4$

$$\begin{aligned}&= -r_4 r_3 r_2 \cdot (((1-r_2)\lambda_1 + (1-r_2(1-s_2))\lambda_2) \wedge (\lambda_3 + (1-s_4)\lambda_4))\\ &\cdot (\lambda_1 \wedge (\lambda_2 + (1-s_3)\lambda_3 + (1-s_3)(1-s_4)\lambda_4))\\ &\cdot ((1-r_3)(1-r_2)\lambda_1 - (1-r_3)(r_2(1-s_2)-1)\lambda_2 + (s_3 + (1-s_3)(1-r_3)(1-r_2(1-s_2)))\lambda_3) \wedge \lambda_4\end{aligned}$$

The constant term (when $\mathfrak{z}_{\vec{e}} = 0$) is

$$\mu_{\{o,1,2,3,4\}}\left(P_{4o}P_{43}P_{3o}P_{32}P_{1o}P_{12}P_{2o}d\mathbf{z}^{\blacklozenge}_{\{o,1,2,3,4\}}\right)$$

$= \int_{([0,1]\times[0,1])^3}$ **G** $\cdot\, dr_2 ds_2 dr_3 ds_3 dr_4 ds_4 \cdot d\mathbf{z}^{\blacklozenge}_o$

$$= F(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \cdot d\mathbf{z}^{\blacklozenge}_o.$$

The polynomial $F(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ can be computed using computer and matches the computation in [4].

4.7. **Proof of theorem 4.13.** Now we turn to the proof of theorem 4.13. To begin, we start with the following lemma.

Lemma 4.15. *We have the following integral formula for the product of derivatives of* $x^i_{12}(\mathfrak{z})$ *and* $P_{12}(\mathfrak{w})$

$$\begin{aligned}&G(\partial_{z_1^1}, \partial_{z_1^2}) x^i_{12}(\mathfrak{z}) \cdot \left(F(\partial_{z_1^1}, \partial_{z_1^2}) \cdot P_{12}(\mathfrak{w})\right)\\ &= G(\partial_{\mathfrak{z}^1}, \partial_{\mathfrak{z}^2}) F(\partial_{\mathfrak{w}^1}, \partial_{\mathfrak{w}^2})(-\partial_{\mathfrak{w}^i}) \int_0^1 P_{12}\left((1-s)\mathfrak{z} + s\mathfrak{w}\right) ds\\ &= \int_0^1 G\left((1-s)\partial_{z_1^1}, (1-s)\partial_{z_1^2}\right) F\left(s\partial_{z_1^1}, s\partial_{z_1^2}\right)(-s\partial_{z_1^i}) P_{12}\left((1-s)\mathfrak{z} + s\mathfrak{w}\right) ds,\end{aligned}$$

where F, G *are arbitrary polynomials in two variables.*

Proof. The lemma follows from the following direct computation

$$\begin{aligned}
&G(\partial_{z_1^1},\partial_{z_1^2})x^i_{12}(\mathfrak{z})\cdot\left(F(\partial_{z_1^1},\partial_{z_1^2})\cdot P_{12}(\mathfrak{w})\right)\\
&=G(\partial_{\mathfrak{z}^1},\partial_{\mathfrak{z}^2})F(\partial_{\mathfrak{w}^1},\partial_{\mathfrak{w}^2})\left(x^i_{12}(\mathfrak{z})P_{12}(\mathfrak{w})\right)\\
&=G(\partial_{\mathfrak{z}^1},\partial_{\mathfrak{z}^2})F(\partial_{\mathfrak{w}^1},\partial_{\mathfrak{w}^2})\left(\frac{x^i_{12}(\mathfrak{z})P_{12}(\mathfrak{z})}{(1+(x_{12}(\mathfrak{z})|\mathfrak{w}-\mathfrak{z}))^2}\right)\\
&=G(\partial_{\mathfrak{z}^1},\partial_{\mathfrak{z}^2})F(\partial_{\mathfrak{w}^1},\partial_{\mathfrak{w}^2})\sum_{m=0}^{\infty}(-x_{12}(\mathfrak{z})|\mathfrak{w}-\mathfrak{z})^m(m+1)x^i_{12}(\mathfrak{z})P_{12}(\mathfrak{z})\\
&=G(\partial_{\mathfrak{z}^1},\partial_{\mathfrak{z}^2})F(\partial_{\mathfrak{w}^1},\partial_{\mathfrak{w}^2})\sum_{m=0}^{\infty}(\partial_{z_1}|\mathfrak{w}-\mathfrak{z})^m\frac{-\partial_{z_1^1}P_{12}(\mathfrak{z})}{m!\cdot(m+2)}\\
&=G(\partial_{\mathfrak{z}^1},\partial_{\mathfrak{z}^2})F(\partial_{\mathfrak{w}^1},\partial_{\mathfrak{w}^2})\int_0^1 e^{s(\partial_{z_1}|\mathfrak{w}-\mathfrak{z})}s\cdot(-\partial_{z_1^i})P_{12}(\mathfrak{z})ds\\
&=G(\partial_{\mathfrak{z}^1},\partial_{\mathfrak{z}^2})F(\partial_{\mathfrak{w}^1},\partial_{\mathfrak{w}^2})(-\partial_{\mathfrak{w}^i})\int_0^1 P_{12}\left((1-s)\mathfrak{z}+s\mathfrak{w}\right)ds\\
&=\int_0^1 G\left((1-s)\partial_{z_1^1},(1-s)\partial_{z_1^2}\right)F\left(s\partial_{z_1^1},s\partial_{z_1^2}\right)(-s\partial_{z_1^i})P_{12}\left((1-s)\mathfrak{z}+s\mathfrak{w}\right)ds.
\end{aligned}$$

□

The next lemma is an immediate consequence of the above formula. It relates the weights of Feynman diagrams when one modifies an edge by multiplying by a coordinate function.

Lemma 4.16. *Suppose that we have a graph with vertices* $\{o,1,\dots,n\}$ *and let* $i,j\in\{o,1,\dots,n\}$ *be vertices such that* $(i,j)\in E_\Gamma$ *is an edge. Then*

$$\mu_{\{o,1,\dots,n\}}\left(W_\Gamma(\mathfrak{z}_e)\cdot x^l_{ij}\right)=-\partial_{\mathfrak{z}^l_{ij}}\int_0^1 F(\lambda_1,\dots,\lambda_n;\dots,r\mathfrak{z}_{ij},\dots)dr,$$

where $F(\lambda_1,\dots,\lambda_n;\mathfrak{z}_e):=\mu_{\{o,1,\dots,n\}}\left(W_\Gamma(\mathfrak{z}_e)\right)$

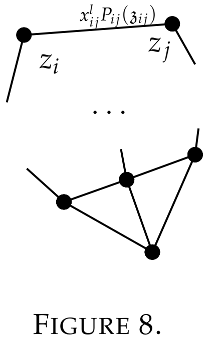


FIGURE 8.

Proof. Using Lemma 4.15, we have

$$x^l_{ij}P_{ij}(\mathfrak{z}_{ij}) = -\partial_{\mathfrak{z}^l_{ij}}\int_0^1 P_{ij}(r\mathfrak{z}_{ij})dr.$$

Then

$$\begin{aligned}\mu_{\{o,1,\dots,n\}}\left(W_\Gamma(\mathfrak{z}_e)\cdot x^l_{ij}\right) &= \mu_{\{o,1,\dots,n\}}\left(-\partial_{\mathfrak{z}^l_{ij}}\int_0^1 W_\Gamma(\dots,r\mathfrak{z}_{ij},\dots)dr\right)\\ &= -\partial_{\mathfrak{z}^l_{ij}}\int_0^1 \mu_{\{o,1,\dots,n\}}\left(W_\Gamma(\dots,r\mathfrak{z}_{ij},\dots)\right)dr\\ &= -\partial_{\mathfrak{z}^l_{ij}}\int_0^1 F(\lambda_1,\dots,\lambda_n;\dots,r\mathfrak{z}_{ij},\dots)dr.\end{aligned}$$

□

Lemma 4.17. *Suppose that we have a graph with vertices* $\{o,1,\dots,n\}$ *and* $i,j\in\{o,1,\dots,n\}$ *such that* $(i,j)\in E_\Gamma$. *Set* $\mu_{\{o,1,\dots,n\}}(\alpha) = F_\alpha(\lambda)$. *Then*

$$\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\dots,n\}}\left(\alpha\cdot x^l_{\star i}P_{\star j}\right) = \int_0^1 F_\alpha(\lambda_1,\dots,\lambda_i+(1-s_\star)\lambda_\star,\dots,\lambda_j+s_\star\lambda_\star,\lambda_n;\mathfrak{z}_e)\cdot s_\star\lambda^l_\star ds_\star$$

$$\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\dots,n\}}\left(\alpha\cdot x^l_{\star j}P_{\star i}\right) = \int_0^1 F_\alpha(\lambda_1,\dots,\lambda_i+(1-s_\star)\lambda_\star,\dots,\lambda_j+s_\star\lambda_\star,\lambda_n;\mathfrak{z}_e)\cdot(1-s_\star)\lambda^l_\star ds_\star$$

Similarly, if $\star$ *is joint with* $(o,i)\in E_\Gamma$, *then*

$$\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\dots,n\}}\left(\alpha\cdot x^l_{\star i}P_{\star o}\right) = \int_0^1 F_\alpha(\lambda_1,\dots,\lambda_i+(1-s_\star)\lambda_\star,\dots,\lambda_n;\mathfrak{z}_e)\cdot s_\star\lambda^l_\star ds_\star$$

$$\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\dots,n\}}\left(\alpha\cdot x^l_{\star o}P_{\star i}\right) = \int_0^1 F_\alpha(\lambda_1,\dots,\lambda_i+(1-s_\star)\lambda_\star,\dots,\lambda_n;\mathfrak{z}_e)\cdot(1-s_\star)\lambda^l_\star ds_\star$$

See Fig. 9 for an illustration of $\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\dots,n\}}\left(\alpha\cdot x^l_{\star i}P_{\star j}\right)$.

Proof. Write

$$\mu_{\{o,1,\dots,n\}}(\alpha) = F_\alpha(\lambda) = \sum G(\lambda_i)H(\lambda_j),$$

where we formally separate the variables λ_i,λ_j. Then

$$\begin{aligned}&\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\dots,n\}}\left(\alpha\cdot x^l_{\star i}P_{\star j}\right) = \mu_{\{\star,\bullet\}}\left(\sum G(\lambda_i-\partial_{z_\star})x^l_{\star\bullet}\cdot H(\lambda_j-\partial_{z_\star})P_{\star\bullet}\right)\\ &= \mu_{\{\star,\bullet\}}\left(\int_0^1\sum G(\lambda_i-(1-s)\partial_{z_\star})H(\lambda_j-s\partial_{z_\star})(-s\partial_{z^l_\star})P_{\star\bullet}ds\right) \quad \text{using Lemma 4.15}\\ &= \int_0^1\sum G(\lambda_i+(1-s)\lambda_\star)H(\lambda_j+s\lambda_\star)(s\lambda^l_\star)ds\\ &= \int_0^1 F_\alpha(\lambda_1,\dots,\lambda_i+(1-s_\star)\lambda_\star,\dots,\lambda_j+s_\star\lambda_\star,\lambda_n;\mathfrak{z}_e)\cdot(s_\star\lambda^l_\star)ds_\star.\end{aligned}$$

□

Now we are ready to prove theorem 4.13.

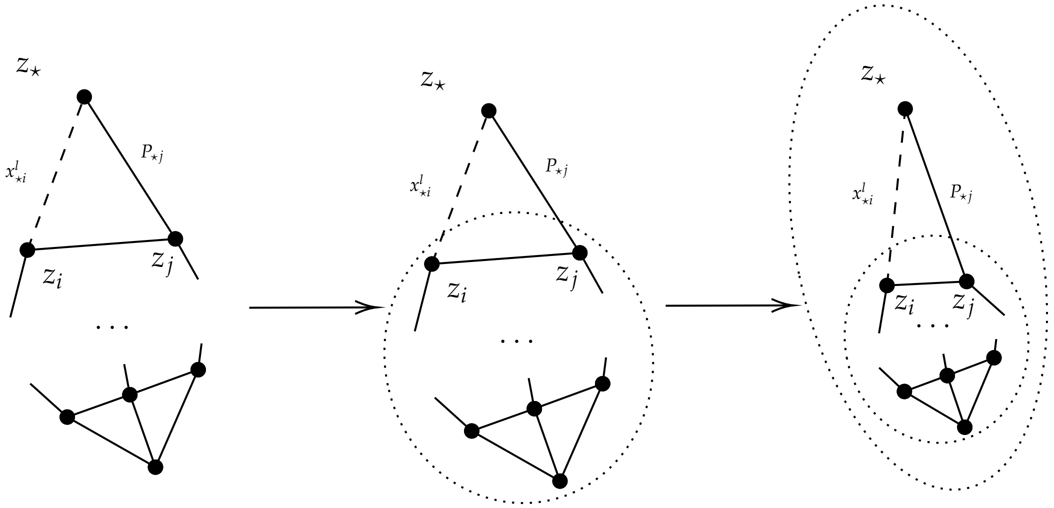


FIGURE 9. $\mu_{\{\star,\bullet\}} \circ \mu_{\{o,1,\dots,n\}} \left(\alpha \cdot x^l_{\star i} P_{\star j}\right)$

Proof. We will only prove the formula in the case when $\star$ is joint with $i, j \in V(\Gamma) - \{o\}$ as the proof for the other case is similar. We first notice that

$$\begin{aligned}\mu_{\{o,1,\dots,n,\star\}}\left(W_{\Gamma^\star}(\mathfrak{z}_e)\right) &= \mu_{\{o,1,\dots,n,\star\}}\left(W_\Gamma(\mathfrak{z}_e) \cdot P_{\star i}(\mathfrak{z}_{\star i}) P_{\star j}(\mathfrak{z}_{\star j})\right) \\ &= \mu_{\{o,1,\dots,n,\star\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e) \cdot P_{\star i} P_{\star j}\right) \cdot e^{(\lambda_i|\mathfrak{z}_{i\star}) + (\lambda_j|\mathfrak{z}_{j\star})}\end{aligned}$$

here

$$\begin{cases} \tilde{\mathfrak{z}}_e = \mathfrak{z}_e + \mathfrak{z}_{\star i}, & \text{if } i \in V(e), e \neq ij, \\ \tilde{\mathfrak{z}}_e = \mathfrak{z}_e + \mathfrak{z}_{\star j}, & \text{if } j \in V(e), e \neq ij, \\ \tilde{\mathfrak{z}}_e = \mathfrak{z}_e - \mathfrak{z}_{\star i} - \mathfrak{z}_{\star j}, & \text{if } e = ij, \\ \tilde{\mathfrak{z}}_e = \mathfrak{z}_e, & \text{otherwise.} \end{cases}$$

Using Lemma 4.7, we have

$$P_{\star i} P_{\star j} P_{ij}(\tilde{\mathfrak{z}}_{ij}) = \mathbf{d}\left(P_{ij}(\tilde{\mathfrak{z}}_{ij}) P_{\star j} \cdot x_{\star i} \wedge x_{ij} + P_{ij}(\tilde{\mathfrak{z}}_{ij}) P_{\star i} \cdot x_{\star j} \wedge x_{ij}\right)$$

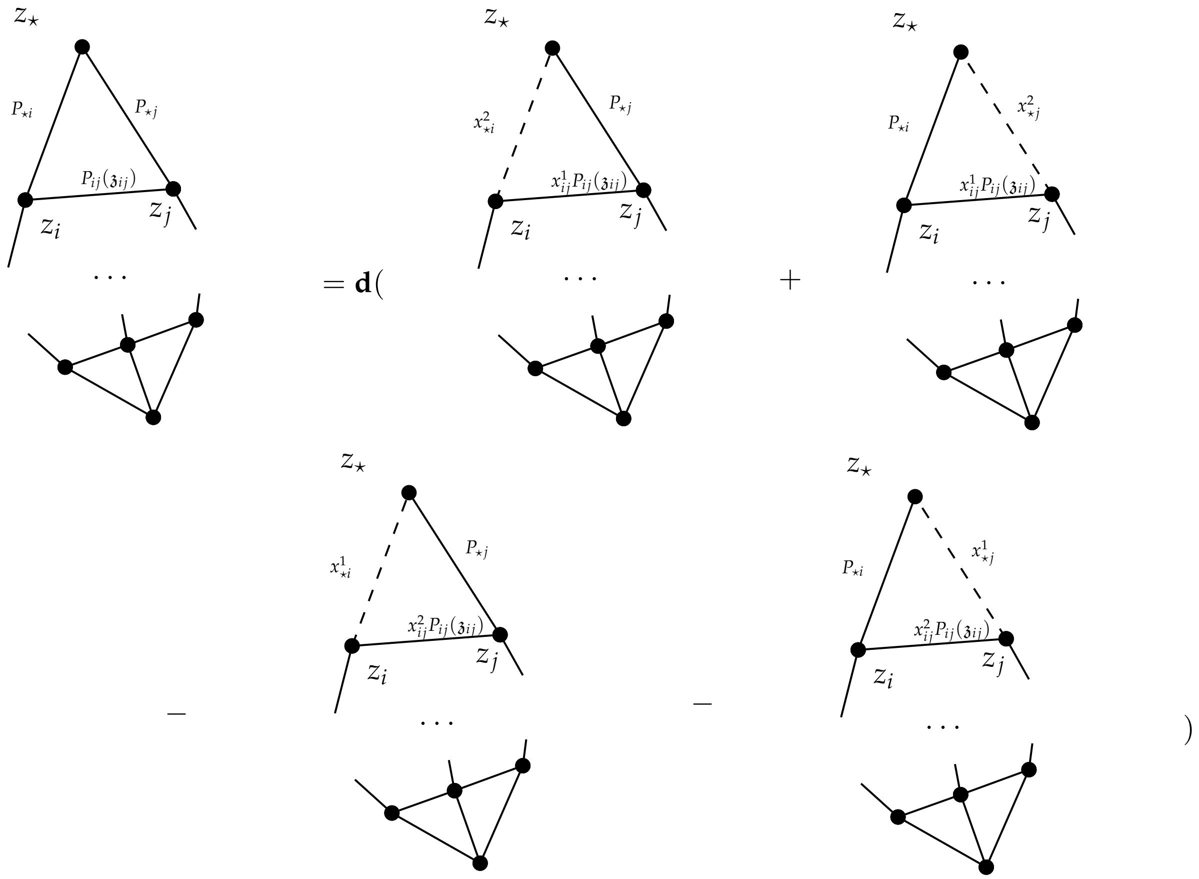

$z_\star$
$P_{\star i}$
$P_{\star j}$
$P_{ij}(\mathfrak{z}_{ij})$
z_i
z_j
$\cdots$
$= \mathbf{d}($
$x^2_{\star i}$
$x^1_{ij}P_{ij}(\mathfrak{z}_{ij})$
$+$
$x^2_{\star j}$
$-$
$x^1_{\star i}$
$x^2_{ij}P_{ij}(\mathfrak{z}_{ij})$
$x^1_{\star j}$
$)$

FIGURE 10. $W_\Gamma(\tilde{\mathfrak{z}}_e) \cdot P_{\star i}P_{\star j}$ is exact by the Arnold relation

Substituting this into $\mu_{\{o,1,\ldots,n,\star\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e) \cdot P_{\star i}P_{\star j}\right)$, we obtain (see Fig. 10 and Fig. 11)

$$\begin{aligned}
&\mu_{\{o,1,\ldots,n,\star\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e) \cdot P_{\star i}P_{\star j}\right)\\
&= \mu_{\{o,1,\ldots,n,\star\}}\left(\mathbf{d}\left(W_\Gamma(\tilde{\mathfrak{z}}_e)P_{\star j} \cdot x_{\star i} \wedge x_{ij} + W_\Gamma(\tilde{\mathfrak{z}}_e)P_{\star i} \cdot x_{\star j} \wedge x_{ij}\right)\right)\\
&= \mu_{\{\star,\bullet\}} \circ \mu_{\{o,1,\ldots,n\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e) \cdot (x_{ij} \wedge x_{\star j})P_{\star i}\right) + \mu_{\{\star,\bullet\}} \circ \mu_{\{1\ldots n\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e) \cdot (x_{ij} \wedge x_{\star i})P_{\star j}\right)\\
&+ \underbrace{\mu_{\{o,1,\ldots,n\}} \circ \mu_{\{\star,i\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e) \cdot (x_{ij} \wedge x_{\star j})P_{\star i}\right)}_{=0} + \underbrace{\mu_{\{o,1,\ldots,n\}} \circ \mu_{\{\star,j\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e) \cdot (x_{ij} \wedge x_{\star i})P_{\star j}\right)}_{=0}.
\end{aligned}$$

We will prove a stronger statement by induction. In fact, we will prove that

$$\sum_{\vec{e}, v \in \vec{e}} \partial_{\mathfrak{z}_{\vec{e}}} \mathbf{W}^{r,s}_{\Gamma, \mathbf{H}_\Gamma(o,e)} = -\lambda_v.$$

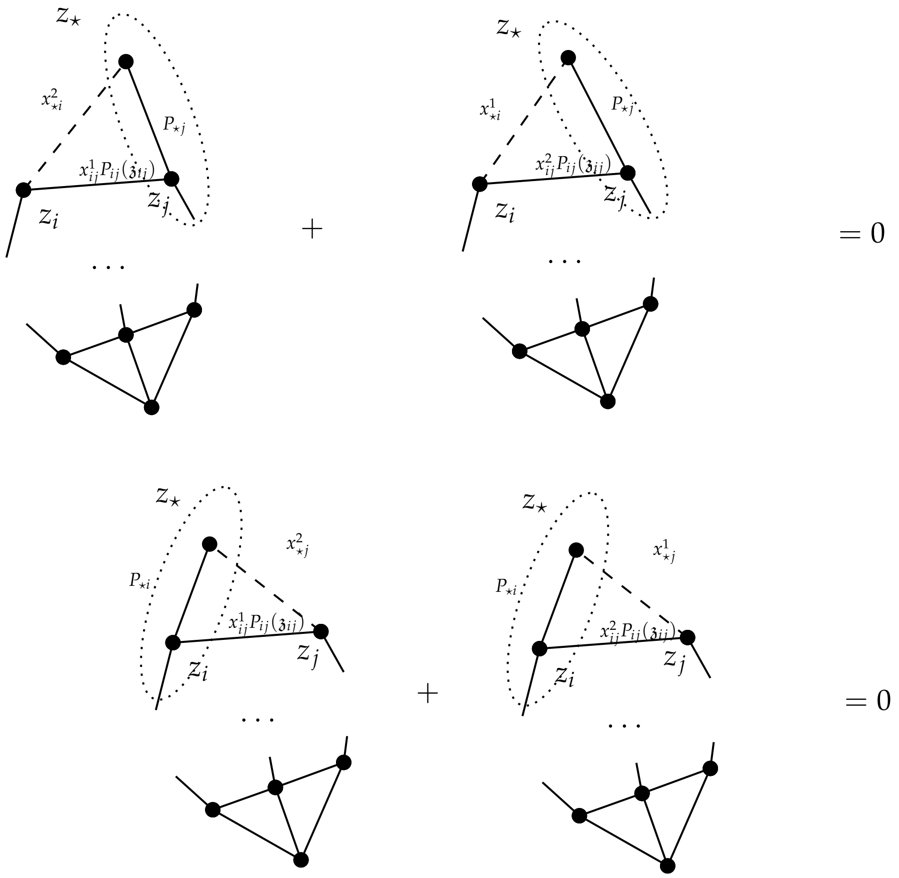

$z_\star$
$x^2_{\star i}$
$P_{\star j}$
$x^1_{ij}P_{ij}(\mathfrak{z}_{ij})$
z_i
z_j
…
+
$z_\star$
$x^1_{\star i}$
$P_{\star j}$
$x^2_{ij}P_{ij}(\mathfrak{z}_{ij})$
z_i
z_j
…
$= 0$
$z_\star$
$P_{\star i}$
$x^2_{\star j}$
$x^1_{ij}P_{ij}(\mathfrak{z}_{ij})$
z_i
z_j
…
+
$z_\star$
$P_{\star i}$
$x^1_{\star j}$
$x^2_{ij}P_{ij}(\mathfrak{z}_{ij})$
z_i
z_j
…
$= 0$

FIGURE 11. Some cancellations

Assume the above is true for $\mathbf{H}_\Gamma(o,e)$, we have

$$
\begin{aligned}
&\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda;\ldots,r_\star\tilde{\mathfrak{z}}_{ij},\ldots)\\
&=\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda;\tilde{\mathfrak{z}}_{\vec{e}})-(1-r_\star)\left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)}(\lambda;\mathfrak{z}_{\vec{e}})|\tilde{\mathfrak{z}}_{ij}\right)\\
&=\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}+\left(\sum_{\vec{e},i\in\vec{e}}\partial_{\mathfrak{z}_{\vec{e}}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}|\mathfrak{z}_{\star i}\right)+\left(\sum_{\vec{e},j\in\vec{e}}\partial_{\mathfrak{z}_{\vec{e}}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}|\mathfrak{z}_{\star j}\right)+(1-r_\star)\left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}|\mathfrak{z}_{\star j}-\mathfrak{z}_{\star i}-\mathfrak{z}_{ij}\right)\\
&=\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}-(\lambda_i|\mathfrak{z}_{\star i})-\left(\lambda_j|\mathfrak{z}_{\star j}\right)+(1-r_\star)\left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}|\mathfrak{z}_{\star j}-\mathfrak{z}_{\star i}-\mathfrak{z}_{ij}\right)\\
&=\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda;\mathfrak{z}^{\triangleright}_{\vec{e}})-(\lambda_i|\mathfrak{z}_{\star i})-\left(\lambda_j|\mathfrak{z}_{\star j}\right).
\end{aligned}
$$

For simplicity of notation, we write $\int_{\square_{\mathbf{H}_\Gamma(o,e)}}$ instead of $\int_{([0,1]\times[0,1])^{|\mathbf{H}_\Gamma(o,e)|}} \bigwedge_{k=1,\ldots,|\mathbf{H}_\Gamma(o,e)|}(dr_k \wedge ds_k)$. By Lemma 4.16, we have (for $l=1,2$)

$$\begin{aligned}
&\mu_{\{o,1,\ldots,n\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e)\cdot x^s_{ij}\right)\\
&= -\partial_{\tilde{\mathfrak{z}}^l_{ij}} \int_0^1 \int_{\square_{\mathbf{H}_\Gamma(o,e)}} e^{\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)}(\lambda;\ldots,r_\star\tilde{\mathfrak{z}}_{ij},\ldots)} \cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}\cdot dr_\star\\
&= -\int_0^1 \int_{\square_{\mathbf{H}_\Gamma(o,e)}} e^{\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)}(\lambda;\ldots,r_\star\tilde{\mathfrak{z}}_{ij},\ldots)} \partial_{\mathfrak{z}^l_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)}\cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}\cdot r_\star dr_\star\\
&= -\int_0^1 \int_{\square_{\mathbf{H}_\Gamma(o,e)}} e^{\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}(o,e)}(\lambda;\mathfrak{z}^{\triangleright}_{\vec{e}})-(\lambda_i|\mathfrak{z}_{\star i})-\left(\lambda_j|\mathfrak{z}_{\star j}\right)} \cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}\cdot \partial_{\mathfrak{z}^l_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}\cdot r_\star dr_\star .
\end{aligned}$$

Now we Lemma 4.17 to get

$$\begin{aligned}
&\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\ldots,n\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e)\cdot(x_{ij}\wedge x_{\star i})P_{\star j}\right)\\
&= \int_0^1\int_0^1\int_{\square_{\mathbf{H}_\Gamma(o,e)}} e^{\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright};\mathfrak{z}^{\triangleright}_{\vec{e}})-\left(\lambda^{\triangleright}_i|\mathfrak{z}_{\star i}\right)-\left(\lambda^{\triangleright}_j|\mathfrak{z}_{\star j}\right)} \cdot \mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright})\cdot\left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright})\wedge\lambda_\star\right)\cdot r_\star s_\star dr_\star\wedge ds_\star\\
&= \int_{([0,1]\times[0,1])^{|\mathbf{H}_\Gamma(o,e)|+1}} e^{\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright};\mathfrak{z}^{\triangleright}_{\vec{e}})-(\lambda_i|\mathfrak{z}_{\star i})-\left(\lambda_j|\mathfrak{z}_{\star j}\right)-\left((1-s_\star)\mathfrak{z}_{\star i}+s_\star\mathfrak{z}_{\star j}|\lambda_\star\right)}\cdot\left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}\wedge\lambda_\star\right)\cdot r_\star\cdot s_\star\\
&\cdot\mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda_1,\ldots,\lambda_i+(1-s_\star)\lambda_\star,\ldots,\lambda_j+s_\star\lambda_\star,\ldots,\lambda_n)\cdot(dr_\star\wedge ds_\star)\cdot\bigwedge_{k=1,\ldots,|\mathbf{H}_\Gamma(o,e)|}(dr_k\wedge ds_k)
\end{aligned}$$

Here we use $\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright})\wedge\lambda_\star = \partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda)\wedge\lambda_\star$ as $\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}$ is a linear function in λ. Similarly, we have

$$\begin{aligned}
&\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\ldots,n\}}\left(W_\Gamma(\mathfrak{z}_e)\cdot(x_{ij}\wedge x_{\star j})P_{\star i}\right)\\
&= \int_{([0,1]\times[0,1])^{|\mathbf{H}_\Gamma(o,e)|+1}} e^{\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright};\mathfrak{z}^{\triangleright}_{\vec{e}})-(\lambda_i|\mathfrak{z}_{\star i})-\left(\lambda_j|\mathfrak{z}_{\star j}\right)-\left((1-s_\star)\mathfrak{z}_{\star i}+s_\star\mathfrak{z}_{\star j}|\lambda_\star\right)}\cdot\left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}\wedge\lambda_\star\right)\cdot r_\star\cdot(1-s_\star)\\
&\cdot\mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda_1,\ldots,\lambda_i+(1-s_\star)\lambda_\star,\ldots,\lambda_j+s_\star\lambda_\star,\ldots,\lambda_n)\cdot(dr_\star\wedge ds_\star)\cdot\bigwedge_{k=1,\ldots,|\mathbf{H}_\Gamma(o,e)|}(dr_k\wedge ds_k).
\end{aligned}$$

We get the final expression

$$\begin{aligned}
&\mu_{\{o,1,\ldots,n,\star\}}\left(W_{\Gamma^\star}(\mathfrak{z}_e)\right)\\
&= \mu_{\{o,1,\ldots,n,\star\}}\left(W_\Gamma(\tilde{\mathfrak{z}}_e)\cdot P_{\star i}P_{\star j}\right)\cdot e^{(\lambda_i|\mathfrak{z}_{i\star})+(\lambda_j|\mathfrak{z}_{j\star})}\\
&= \left(\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\ldots,n\}}\left(W_\Gamma(\mathfrak{z}_e)\cdot(x_{ij}\wedge x_{\star j})P_{\star i}\right)+\mu_{\{\star,\bullet\}}\circ\mu_{\{o,1,\ldots,n\}}\left(W_\Gamma(\mathfrak{z}_e)\cdot(x_{ij}\wedge x_{\star i})P_{\star j}\right)\right)\cdot e^{(\lambda_i|\mathfrak{z}_{i\star})+(\lambda_j|\mathfrak{z}_{j\star})}\\
&= \int_{([0,1]\times[0,1])^{|\mathbf{H}_\Gamma(o,e)|+1}} e^{\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda^{\triangleright};\mathfrak{z}^{\triangleright}_{\vec{e}})-\left((1-s_\star)\mathfrak{z}_{\star i}+s_\star\mathfrak{z}_{\star j}|\lambda_\star\right)}\cdot\left(\partial_{\mathfrak{z}_{ij}}\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}\wedge\lambda_\star\right)\cdot r_\star\\
&\cdot\mathbf{G}^{r,s}_{\Gamma,\mathbf{H}_\Gamma(o,e)}(\lambda_1,\ldots,\lambda_i+(1-s_\star)\lambda_\star,\ldots,\lambda_j+s_\star\lambda_\star,\ldots,\lambda_n)\cdot(dr_\star\wedge ds_\star)\cdot\bigwedge_{k=1,\ldots,|\mathbf{H}_\Gamma(o,e)|}(dr_k\wedge ds_k)
\end{aligned}$$

We finish the induction step by showing that

$$\sum_{\vec{e}\in E(\Gamma^\star),i\in\vec{e}} \partial_{\mathfrak{z}_{\vec{e}}} \mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} = -\lambda_i, \quad \sum_{\vec{e}\in E(\Gamma^\star),j\in\vec{e}} \partial_{\mathfrak{z}_{\vec{e}}} \mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} = -\lambda_j, \quad \sum_{\vec{e}\in E(\Gamma^\star),\star\in\vec{e}} \partial_{\mathfrak{z}_{\vec{e}}} \mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} = -\lambda_\star,$$

here

$$\mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} = \mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_{\Gamma}(o,e)}(\lambda^{\triangleright};\mathfrak{z}^{\triangleright}_{\vec{e}}) - \left(\lambda_\star|(1-s_\star)\mathfrak{z}_{\star i} + s_\star\mathfrak{z}_{\star j}\right).$$

We can write

$$\begin{aligned}\mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} = {} & \left(1 + (\lambda_\star|(1-s_\star)\partial_{\lambda_i} + s_\star\partial_{\lambda_j})\right)\left(1 + (1-r_\star)(\mathfrak{z}_{\star j} - \mathfrak{z}_{\star i} - \mathfrak{z}_{ij}|\partial_{\mathfrak{z}_{ij}})\right)\mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_{\Gamma}(o,e)} \\ & - \left(\lambda_\star|(1-s_\star)\mathfrak{z}_{\star i} + s_\star\mathfrak{z}_{\star j}\right)\end{aligned}$$

Then we can compute

$$\begin{aligned}& \sum_{\vec{e}\in E(\Gamma^\star),i\in\vec{e}} \partial_{\mathfrak{z}_{\vec{e}}} \mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} \\ & = \left(1 + (\lambda_\star|(1-s_\star)\partial_{\lambda_i} + s_\star\partial_{\lambda_j})\right)\left(1 + (1-r_\star)(\mathfrak{z}_{\star j} - \mathfrak{z}_{\star i} - \mathfrak{z}_{ij}|\partial_{\mathfrak{z}_{ij}})\right)\sum_{\vec{e}\in E(\Gamma),i\in\vec{e}} \partial_{\mathfrak{z}_{\vec{e}}} \mathbf{W}^{r,s}_{\Gamma,\mathbf{H}_{\Gamma}(o,e)} + (1-s_\star)\lambda_\star \\ & = \left(1 + (\lambda_\star|(1-s_\star)\partial_{\lambda_i} + s_\star\partial_{\lambda_j})\right)(-\lambda_i) + (1-s_\star)\lambda_\star \\ & = -\lambda_i.\end{aligned}$$

We have similar computation for $\sum_{\vec{e}\in E(\Gamma^\star),j\in\vec{e}} \partial_{\mathfrak{z}_{\vec{e}}} \mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} = -\lambda_j$. And

$$\sum_{\vec{e}\in E(\Gamma^\star),\star\in\vec{e}} \partial_{\mathfrak{z}_{\vec{e}}} \mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} = (\partial_{\mathfrak{z}_{\star i}} + \partial_{\mathfrak{z}_{\star j}})\mathbf{W}^{r,s}_{\Gamma^\star,\mathbf{H}_{\Gamma^\star}(o,e)} = -\lambda_\star.$$

The proof is completed. □

APPENDIX A. GRAPH THEORY BACKGROUND

In this appendix, we introduce some concepts and facts from graph theory used in section 2.

Definition A.1. A directed graph $\vec{\Gamma}$ consists of the following data:

(1) An ordered set $\vec{\Gamma}_0$ of vertices. We use $|\Gamma_0|$ to denote the number of vertices.
(2) An ordered set $\vec{\Gamma}_0$ of directed edges. We use $|\Gamma_1|$ to denote the number of directed edges.
(3) Two maps

$$t, h : \Gamma_1 \to \Gamma_0,$$

which are assignments of tail and head to each directed edge. We require that

$$t(e) \neq h(e)$$

for any $e \in \vec{\Gamma}_1$, i.e., the graph $\vec{\Gamma}$ has no self-loops.

Furthermore, we say $\vec{\Gamma}$ is decorated (Feynman) if we have a special edge $e_l \in \vec{\Gamma}_1$ and a map

$$m : \Gamma_1 \to \{1, 2, \ldots, d\}.$$

Definition A.2. Let $\vec{\Gamma}$ be a directed graph. The incidence matrix $\rho = (\rho_{ei})_{e \in \vec{\Gamma}_1, i \in \vec{\Gamma}_0}$ is given by

$$\rho_{ei} = \begin{cases} 1, & \text{if } t(e) = i, \\ -1, & \text{if } s(e) = i, \\ 0, & \text{otherwise.} \end{cases}$$

Definition A.3. Given a connected directed graph $\vec{\Gamma}$, a tree $T \subset \vec{\Gamma}$ is said to be a spanning tree if every vertex of $\vec{\Gamma}$ lies in T. We denote the set of all spanning tree by $\mathrm{Tree}(\vec{\Gamma})$.

Definition A.4. Given a connected directed graph $\vec{\Gamma}$ and two disjoint subsets of vertices $V_1,\ V_2 \subset \vec{\Gamma}_0$, we define $\mathrm{Cut}(\vec{\Gamma}; V_1, V_2)$ to be the set of subsets $C \subset \vec{\Gamma}_1$ satisfying the following properties:

1. (1) The removing of edges in C from $\vec{\Gamma}$ divides $\vec{\Gamma}$ into exactly two connected trees, which we denoted by $\vec{\Gamma}'(C),\ \vec{\Gamma}''(C)$, such that $V_1 \subset \vec{\Gamma}_0'(C),\ V_2 \subset \vec{\Gamma}_0''(C)$.
2. (2) C doesn't contain any proper subset satisfying property 1.

Definition A.5. Given a connected directed graph $\vec{\Gamma}$, and a function maps each $e \in \vec{\Gamma}_1$ to $t_e \in (0, +\infty)$, we define the weighted laplacian of $\vec{\Gamma}$ by the following formula:

$$M_{\vec{\Gamma}}(t)_{ij} = \sum_{e \in \vec{\Gamma}_1} \rho_i^e \frac{1}{t_e} \rho_j^e, \quad 1 \leqslant i, j \leqslant |\Gamma_0| - 1.$$

The following facts will be used to show the finitenes of Feynman graph integrals.

Theorem A.6 (Kirchhoff). *Given a connected graph $\vec{\Gamma}$, the determinant of weighted laplacian is given by the following formula:*

$$\det M_{\vec{\Gamma}}(t) = \frac{\sum\limits_{T \in \mathrm{Tree}(\vec{\Gamma})} \prod\limits_{e \notin T} t_e}{\prod\limits_{e \in \vec{\Gamma}_1} t_e}$$

Corollary A.7. *The inverse of $M_{\vec{\Gamma}}(t)$ is given by the following formula:*

$$M_{\vec{\Gamma}}(t)_{ij}^{-1} = \frac{1}{\sum\limits_{T \in \mathrm{Tree}(\vec{\Gamma})} \prod\limits_{e \notin T} t_e} \cdot \left(\sum_{C \in \mathrm{Cut}(\vec{\Gamma}; \{i,j\}, \{|\Gamma_0|\})} \prod_{e \in C} t_e \right)$$

Definition A.8. Let $\vec{\Gamma}$ be a connected directed graph. For any $e \in \vec{\Gamma}_1$, $t_e \in (0, +\infty)$. The graphic Green's function $d_{\vec{\Gamma}}^{-1}(t) = (d_{\vec{\Gamma}}^{-1}(t)_{ei})_{e\in\vec{\Gamma}_1, i\in\vec{\Gamma}_0-\{|\Gamma_0|\}}$ is given by

$$d_{\vec{\Gamma}}^{-1}(t)_{ei} = \sum_{j=1}^{|\Gamma_0|-1} \frac{1}{t_e}\rho_{ej}M_{\vec{\Gamma}}^{-1}(t)_{ji}.$$

Proposition A.9 (see appendix B of [16])**.** *We have the following equality:*

$$\begin{aligned} & d_{\vec{\Gamma}}^{-1}(t)_{ei} \\ = \; & \frac{1}{\sum\limits_{T\in\mathrm{Tree}(\vec{\Gamma})}\prod\limits_{e\notin T} t_e}\left(\sum_{C\in\mathrm{Cut}(\vec{\Gamma};\{i,t(e)\},\{|\Gamma_0|,h(e)\})} \frac{\prod_{e'\in C} t_{e'}}{t_e}\right. \\ - \; & \left.\sum_{C\in\mathrm{Cut}(\vec{\Gamma};\{i,h(e)\},\{|\Gamma_0|,t(e)\})} \frac{\prod_{e'\in C} t_{e'}}{t_e}\right) \end{aligned}$$

In particular, every term of the numerator also appears in the denominator, so we have

$$\left|d_{\vec{\Gamma}}^{-1}(t)_{ei}\right| \leqslant 2.$$

Remark A.10. If we view graphs as discrete spaces, the incidence matrix can be viewed as de Rham differential. Then $d_{\vec{\Gamma}}^{-1}(t)_{ei}$ can be viewed as the Green's function of de Rham differential on a graph. This might explains the importance of $d_{\vec{\Gamma}}^{-1}(t)_{ei}$.

APPENDIX B. COMPACTIFICATION OF SCHWINGER SPACES

In this appendix, we review the construction of a compactification of Schwinger space in [19, 20].

Definition B.1. Given a directed graph $\vec{\Gamma}$, and $L > 0$, the Schwinger space is defined by $(0, L]^{\vec{\Gamma}_1}$. The orientation is given by the following formula:

$$\int_{(0,L]^{\vec{\Gamma}_1}} \prod_{e\in\vec{\Gamma}_1} dt_e = L^{|\Gamma_1|}.$$

Assume $\vec{\Gamma}$ is a connected directed graph, $L > 0$ is a positive real number. Let $S \subset \vec{\Gamma}_1$ be a subset of $\vec{\Gamma}_1$, we define the following submanifold with corners of Schwinger space:

$$\Delta_S = \left\{(t_1, t_2, \dots, t_{|\Gamma_1|}) \in [0, L]^{\vec{\Gamma}_1} | t_e = 0 \quad \text{if } e \in S\right\}.$$

The compactification of Schwinger space is obtained by iterated real blow ups of $[0, L]^{\vec{\Gamma}_1}$ along Δ_S for all $S \subset \vec{\Gamma}_1$ in a certain order (see [1, 14]). To avoid getting into technical details of real blow ups of manifolds with corners, we will use another simpler definition. Instead, we present a typical example of real blow up, which will be helpful to understand our construction:

Example B.2. Let $S = \{1, 2, \ldots, k\} \subset \vec{\Gamma}_1$, the real blow up of $[0, +\infty)^{\vec{\Gamma}_1}$ along Δ_S is the following manifold(with corners):

$$[[0, +\infty)^{\vec{\Gamma}_1} : \Delta_S] := \left\{ (\rho, \xi_1, \ldots, \xi_k, t_{k+1}, \ldots t_{|\Gamma_1|}) \in [0, +\infty)^{|\Gamma_1|+1} \left| \sum_{i=1}^{k} \xi_i^2 = 1 \right. \right\}.$$

We have a natural map from $[[0, +\infty)^{\vec{\Gamma}_1} : \Delta_S]$ to $[0, +\infty)^{\vec{\Gamma}_1}$:

$$(\rho, \xi_1, \ldots, \xi_k, t_{k+1}, \ldots t_{|\Gamma_1|}) \to (t_1 = \rho\xi_1, \ldots, t_k = \rho\xi_k, t_{k+1}, \ldots, t_{|\Gamma_1|}).$$

We also have a natural inclusion map from $(0, +\infty)^{\vec{\Gamma}_1}$ to $[[0, +\infty)^{\vec{\Gamma}_1} : \Delta_S]$:

$$\begin{aligned} &(t_1, \ldots, t_{|\Gamma_1|}) \\ \to &\left(\rho = \sqrt{\sum_{i=1}^{k} t_i^2}, \xi_1 = \frac{t_1}{\sqrt{\sum_{i=1}^{k} t_i^2}}, \ldots, \xi_k = \frac{t_k}{\sqrt{\sum_{i=1}^{k} t_i^2}}, t_{k+1}, \ldots, t_{|\Gamma_1|} \right). \end{aligned}$$

For us, the most important property is that $\frac{t_i}{\sqrt{\sum_{i=1}^{k} t_i^2}}$ can be extended to a smooth function ξ_i on $[[0, +\infty)^{\vec{\Gamma}_1} : \Delta_S]$.

Let's consider the following natural inclusion map:

$$i : (0, +\infty)^{\vec{\Gamma}_1} \to \prod_{S \subset \vec{\Gamma}_1} [[0, +\infty)^{\vec{\Gamma}_1} : \Delta_S]$$

Definition B.3. We call the closure of the image of $(0, L]^{\vec{\Gamma}_1}$ under i the compactified Schwinger space of $\vec{\Gamma}$. We denote it by $\widetilde{[0, L]}^{\vec{\Gamma}_1}$.

Remark B.4. Sometimes, by abusing of concept, we will also call the closure of the image of $(0, +\infty)^{\vec{\Gamma}_1}$ under i the compactified Schwinger space, although it is not compact. We will use $\widetilde{[0, +\infty)}^{\vec{\Gamma}_1}$ to denote it.

Proposition B.5 ([1])**.** *$\widetilde{[0, L]}^{\vec{\Gamma}_1}$ is a compact manifold with corners.*

The matrices $M_{\vec{\Gamma}}(t)_{ij}^{-1}$ and $d_{\vec{\Gamma}}^{-1}(t)_{ei}$ defined in appendix A admit extensions to matrix-valued smooth functions on $\widetilde{[0, +\infty)}^{|\Gamma_1|}$:

Lemma B.6 ([19])**.** *Given a connected graph $\vec{\Gamma}$, The following functions can be extended to smooth functions on $\widetilde{[0, +\infty)}^{\vec{\Gamma}_1}$:*

(1) $M_{\vec{\Gamma}}(t)^{-1}_{ij}$ *for* $1 \leqslant i, j \leqslant |\Gamma_0| - 1$.
(2) $d^{-1}_{\vec{\Gamma}}(t)_{ei}$ *for* $e \in \Gamma_1$, $1 \leqslant j \leqslant |\Gamma_0| - 1$.

Let $\bar{S}^+((0,+\infty)^{\vec{\Gamma}_1})$ be the closure of

$$S^+((0,+\infty)^{\vec{\Gamma}_1}) = \{(t_{e_1}, \ldots, t_{e_{|\Gamma_1|}}) \in (0,+\infty)^{\vec{\Gamma}_1} | \sum_{i=1}^{|\Gamma_1|} t_i^2 = 1\}$$

in $\widetilde{[0,+\infty)}^{\vec{\Gamma}_1}$, the following proposition give a description of the boundary of $\bar{S}^+((0,+\infty)^{\vec{\Gamma}_1})$:

Proposition B.7. *Given a connected graph* $\vec{\Gamma}$, *the boundary of* $\bar{S}^+((0,+\infty)^{\vec{\Gamma}_1})$ *has the following decomposition:*

$$\partial \bar{S}^+((0,+\infty)^{\vec{\Gamma}_1}) = \bigcup_{\vec{\Gamma}' \subset \vec{\Gamma}} \mathrm{sgn}(\sigma_{\vec{\Gamma}'_1 \subset \vec{\Gamma}_1}) \bar{S}^+((0,+\infty)^{\vec{\Gamma}'_1}) \times \bar{S}^+((0,+\infty)^{\vec{\Gamma}_1 - \vec{\Gamma}'_1}).$$

Proof. This follows from the construction of $\widetilde{[0,+\infty)}^{\vec{\Gamma}_1}$. ☐

APPENDIX C. MATHEMATICA CODE FOR THE CHIRAL 5-OPERATION

Here is the Mathematica code for the chiral 5-operation (compare with [4, Section 4.8 (4.67)]).

```
(*Define wedge*)
wedge[a_, b_] := a[[1]]*b[[2]] - a[[2]]*b[[1]];

{r2, r3, r4, s2, s3, s4} =
  Symbol /@ {"r2", "r3", "r4", "s2", "s3", "s4"};
\[Lambda]1 = {\[Lambda]11, \[Lambda]12};
\[Lambda]2 = {\[Lambda]21, \[Lambda]22};
\[Lambda]3 = {\[Lambda]31, \[Lambda]32};
\[Lambda]4 = {\[Lambda]41, \[Lambda]42};
(*Our formula*)
Integrate[wedge[(1 - r2)*\[Lambda]1 + (1 - r2*(1 - s2))*\[
   Lambda]3,
       \[Lambda]4 + (1 - s4)*\[Lambda]2]*
  wedge[\[Lambda]1, \[Lambda]3 + (1 - s3)*\[Lambda]4 +
         (1 - s3)*(1 - s4)*\[Lambda]2]*
  wedge[(-(1 - r3))*(1 - r2)*\[Lambda]1 +
         (1 - r3)*(r2*(1 - s2) - 1)*\[Lambda]3 +
         (-s3 - (1 - s3)*(1 - r3)*(1 -
          r2*(1 - s2)))*\[Lambda]4, \[Lambda]2]*r2*r3*
     r4, {s4, 0, 1}, {s3, 0, 1}, {s2, 0, 1}, {r4, 0, 1},
   {r3, 0, 1}, {r2, 0, 1}]
(*Result in [4, Section 4.8
(4.67)]*)
F = (1/864)*wedge[\[Lambda]1, \[Lambda]2]^3 + (1/96)*
    wedge[\[Lambda]1, \[Lambda]3]*
    wedge[\[Lambda]2, \[Lambda]3]^2 - (1/96)*
    wedge[\[Lambda]1, \[Lambda]4]^2*
    wedge[\[Lambda]2, \[Lambda]3] + (1/96)*
    wedge[\[Lambda]1, \[Lambda]4]*
    wedge[\[Lambda]2, \[Lambda]3]^2 - (1/96)*
    wedge[\[Lambda]1, \[Lambda]3]*
    wedge[\[Lambda]2, \[Lambda]4]^2 - (1/96)*
    wedge[\[Lambda]1, \[Lambda]4]^2*
    wedge[\[Lambda]2, \[Lambda]4] + (1/96)*
    wedge[\[Lambda]1, \[Lambda]3]^2*
    wedge[\[Lambda]2, \[Lambda]4] - (1/8)*
    wedge[\[Lambda]1, \[Lambda]3]*wedge[\[Lambda]2, \[Lambda
       ]4]*
    wedge[\[Lambda]3, \[Lambda]4] + (1/48)*
    wedge[\[Lambda]1, \[Lambda]4]*wedge[\[Lambda]2, \[Lambda
       ]3]*
    wedge[\[Lambda]2, \[Lambda]4] - (1/48)*
    wedge[\[Lambda]1, \[Lambda]4]*wedge[\[Lambda]2, \[Lambda
       ]4]*
    wedge[\[Lambda]3, \[Lambda]4] - (1/96)*
    wedge[\[Lambda]1, \[Lambda]3]*wedge[\[Lambda]2, \[Lambda
       ]4]*
    wedge[\[Lambda]1, \[Lambda]2] - (1/48)*
    wedge[\[Lambda]1, \[Lambda]3]*wedge[\[Lambd3a]2, \[Lambda
       ]3]*
    wedge[\[Lambda]3, \[Lambda]4] + (1/16)*
```

```
wedge[\[Lambda]1, \[Lambda]4]*wedge[\[Lambda]2, \[Lambda
   ]3]*
wedge[\[Lambda]3, \[Lambda]4] - (1/48)*
wedge[\[Lambda]1, \[Lambda]3]*wedge[\[Lambda]1, \[Lambda
   ]4]*
wedge[\[Lambda]2, \[Lambda]4] + (1/48)*
wedge[\[Lambda]1, \[Lambda]3]*wedge[\[Lambda]2, \[Lambda
   ]3]*
wedge[\[Lambda]2, \[Lambda]4] + (1/144)*
wedge[\[Lambda]1, \[Lambda]3]*wedge[\[Lambda]1, \[Lambda
   ]4]*
wedge[\[Lambda]1, \[Lambda]2] - (1/48)*
wedge[\[Lambda]1, \[Lambda]3]*wedge[\[Lambda]1, \[Lambda
   ]4]*
wedge[\[Lambda]2, \[Lambda]3] - (1/96)*
wedge[\[Lambda]1, \[Lambda]3]*wedge[\[Lambda]2, \[Lambda
   ]3]*
wedge[\[Lambda]1, \[Lambda]2] - (1/96)*
wedge[\[Lambda]1, \[Lambda]4]*wedge[\[Lambda]2, \[Lambda
   ]3]*
wedge[\[Lambda]1, \[Lambda]2] - (1/96)*
wedge[\[Lambda]1, \[Lambda]4]*wedge[\[Lambda]2, \[Lambda
   ]4]*
wedge[\[Lambda]1, \[Lambda]2] + (1/144)*
wedge[\[Lambda]2, \[Lambda]3]*wedge[\[Lambda]2, \[Lambda
   ]4]*
wedge[\[Lambda]1, \[Lambda]2] - (1/288)*
wedge[\[Lambda]2, \[Lambda]4]*
wedge[\[Lambda]1, \[Lambda]2]^2 + (7/96)*
wedge[\[Lambda]3, \[Lambda]4]^2*
wedge[\[Lambda]1, \[Lambda]2] - (1/288)*
wedge[\[Lambda]2, \[Lambda]3]*
wedge[\[Lambda]1, \[Lambda]2]^2 + (1/288)*
wedge[\[Lambda]1, \[Lambda]4]^2*
wedge[\[Lambda]1, \[Lambda]2] + (1/288)*
```

```
wedge[\[Lambda]1, \[Lambda]2]^2*
wedge[\[Lambda]1, \[Lambda]4] + (1/288)*
wedge[\[Lambda]1, \[Lambda]2]^2*
wedge[\[Lambda]1, \[Lambda]3] + (1/288)*
wedge[\[Lambda]2, \[Lambda]3]^2*wedge[\[Lambda]1, \[
   Lambda]2];
```

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