Exceptional symmetry in six-dimensional superconformal quantum field theory

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Abstract. We introduce a dg Lie superalgebra which we conjecture to be present as a symmetry in the holomorphic twist of any six-dimensional $\mathcal{N} = (2,0)$ superconformal QFT. It is closely related to Kac's exceptional Lie superalgebra $E(3|6)$. We justify this expectation by presenting this symmetry in the twist of the abelian superconformal QFT, or "tensor multiplet" theory. Finally, we prove that the further localization of our symmetry algebra to a two-dimensional chiral plane is the universal symmetry algebra of two-dimensional chiral CFT; the Virasoro algebra.

Contents

1 Introduction

(Super)conformal algebras are finite dimensional for spacetimes of dimension $d > 2$. On the other hand, we will provide evidence that the holomorphic twist of a six-dimensional $\mathcal{N} = (2,0)$ superconformal field theory has an infinite-dimensional symmetry akin to the Virasoro algebra in conformal field theory. It is a particular enhancement of the finite-dimensional twisted six-dimensional superconformal algebra. The purpose of this paper is to describe this enhancement and to study its realization in the holomorphic twist of the free $\mathcal{N} = (2,0)$ theory. The enhanced symmetry is governed by a local version of Kac's exceptional linearly compact Lie superalgebra $E(3|6)$. After a further deformation and localization to a complex curve, this exceptional symmetry reduces to the Virasoro symmetry of a chiral conformal field theory.

This gives a six-dimensional counterpart of a familiar feature of two-dimensional conformal field theory. Infinitesimal conformal transformations on a formal complex disk form the Witt algebra, whose quantum central extension is the Virasoro algebra. In the present setting, $E(3|6)$ plays the role of the formal infinitesimal symmetry algebra, while its sheaf-theoretic versions provide the local symmetries required for field theory.

The exceptional local symmetry

Let X be a complex threefold, let $\mathcal{R}$ be a rank-two holomorphic vector bundle on X, and fix an isomorphism

$$\Omega\colon\ \wedge^2 \mathcal{R} \xrightarrow{\cong} \mathcal{K}_X. \tag{1.1}$$

We associate to this data a holomorphic Lie superalgebra whose even and odd parts split as:

$$\mathcal{E}(3|6)_{\mathcal{R}} = \mathcal{A}\mathrm{t}_\Omega(\mathcal{R}) \oplus \Pi\big(\Omega^1_X \otimes \mathcal{R}^*\big). \tag{1.2}$$

Here $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ consists of first-order differential operators on $\mathcal{R}$ whose induced action on $\det(\mathcal{R}) \cong \mathcal{K}_X$ is the Lie derivative along their symbol. It is the even part of $\mathcal{E}(3|6)_{\mathcal{R}}$. The odd part is $\Omega^1_X \otimes \mathcal{R}^*$ (note that $\Pi(-)$ denotes parity shift), and its symmetric bracket is a first-order differential operator in $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$. The Dolbeault resolution

$$\mathscr{E}(3|6)_{\mathcal{R}} = \Omega^{0,\bullet}(X, \mathcal{E}(3|6)_{\mathcal{R}}) \tag{1.3}$$

is a local dg Lie superalgebra in the sense of [CG21].

On the formal three-disk, after trivializing $\mathcal{R}$ compatibly with Ω, formal sections of $\mathcal{E}(3|6)_{\mathcal{R}}$ recover the standard linearly compact Lie superalgebra $E(3|6)$ [Kac98; CK99]. Polynomial sections give its polynomial form, while holomorphic sections give the corresponding analytic version.

Section 3 provides a supergeometric origin for this exceptional Lie superalgebra. The relevant symbol is the two-step nilpotent graded Lie superalgebra

$$\mathfrak{m}_{-2} = A, \qquad \mathfrak{m}_{-1} = \Pi(A^* \otimes B^*), \tag{1.4}$$

where $\dim A = 3$, $\dim B = 2$, and the bracket is determined by an identification

$$\wedge^2 B \cong \wedge^3 A^*.$$

The full Tanaka prolongation of this symbol and its degree-zero derivations is $E(3|6)$. Equivalently, $E(3|6)$ is the algebra of formal infinitesimal automorphisms of the flat rank-$0|6$, bracket-generating distribution on the associated $3|6$-dimensional formal supermanifold. We also construct global examples of this Tanaka geometry, including a homogeneous structure over $\mathbb{P}^3$.

The appearance of $E(3|6)$ after holomorphic twisting is explained by the derived superconformal-symmetry calculation of [Hah+24], recalled in Theorem 3.1: on flat space the twisted local symmetry algebra is quasi-isomorphic to $\mathscr{E}(3|6)_{\mathcal{R}}$. This motivates the expectation that $\mathscr{E}(3|6)_{\mathcal{R}}$ acts on the holomorphic twist of any $\mathcal{N} = (2,0)$ superconformal theory. We do not prove that general statement here. Instead, we construct the action explicitly for the abelian theory.

The free tensor multiplet and its anomaly

The abelian $\mathcal{N} = (2,0)$ theory, or tensor multiplet, is the one six-dimensional example for which the holomorphic twist can be described directly. It is not a conventional Lagrangian theory because of the self-dual tensor field. Following [SW23a], we formulate its twist as a free Poisson-degenerate BV theory.

Our first field-theoretic result is an explicit classical action of the exceptional local symmetry. More precisely, in Theorem 4.10 we construct an L_∞ local current

$$\mathbf{T}\colon \mathscr{E}(3|6)_{\mathcal{R}} \rightsquigarrow \mathbb{O}_{\mathrm{loc}}(\mathscr{V})[-1] \tag{1.5}$$

which satisfies the equivariant classical master equation. Thus the holomorphic twist of the free tensor multiplet carries a classical $\mathscr{E}(3|6)_{\mathcal{R}}$-symmetry. The higher Taylor components of $\mathbf{T}$ are essential: the symmetry is naturally an L_∞-action rather than a strict action by Hamiltonian local functionals.

We then quantize both the observables and the current using factorization algebras. The factorization form of Noether's theorem produces a map

$$\mathbf{T}^q\colon \mathbf{U}_{\Theta_{fr}}(\mathscr{E}(3|6)_{\mathcal{R}}) \longrightarrow \mathrm{Obs}_{\mathcal{R}}, \tag{1.6}$$

where $\mathrm{Obs}_{\mathcal{R}}$ is the factorization algebra of quantum observables and Θ_{fr} is the one-loop obstruction to quantizing the classical current. The source is the factorization enveloping algebra of $\mathscr{E}(3|6)_{\mathcal{R}}$, twisted by this anomaly.

We compute the restriction of Θ_{fr} to the even symmetry algebra by equivariant Riemann–Roch. It is a universal degree-eight polynomial in the Chern classes of T_X and the residual $SU(2)$-bundle. After the characteristic classes are specialized according to the holomorphic twist, the result agrees with the standard anomaly polynomial of the six-dimensional free tensor multiplet. This comparison fixes the normalization of the central extension used later in the localization calculation.

Localization to a curve

The second part of the paper studies a further deformation of the *polarized theory* and provides a factorization algebra refinement of the work of [BRR15] which associates a vertex algebra to a six-dimensional superconformal QFT. By polarized we mean a $\mathbf{Z}$-graded lift of both the theory and symmetry which exists in the following geometric context. Let us suppose that the threefold is a product

$$X = C \times S$$

where C is a complex curve and S a complex surface. The natural choice of a rank two vector bundle for this geometry is

$$\mathcal{R} = \pi_C^*\mathcal{K}_C \oplus \pi_S^*\mathcal{K}_S \tag{1.7}$$

This spliting allows $\mathcal{E}(3|6)_{\mathcal{R}}$ to admit consistent $\mathbb{Z}$-grading. We denote the resulting graded holomorphic Lie algebra by $\widetilde{\mathcal{E}}(3|6)$, and its Dolbeault resolution by $\widetilde{\mathscr{E}}(3|\underset{\sim}{6})$.

A holomorphic vector field v on S determines a Maurer–Cartan element of $\widetilde{\mathscr{E}}(3|6)$. It therefore deforms both the exceptional symmetry algebra and every factorization algebra equipped with the corresponding current. For

$$S = \mathbf{C}^2, \qquad v = \mathsf{Eu} = w_1\frac{\partial}{\partial w_1} + w_2\frac{\partial}{\partial w_2},$$

the deformation is supported at the unique zero of v. Pushing forward along

$$\pi\colon C\times \mathbf{C}^2 \longrightarrow C$$

therefore produces a factorization algebra on the curve.

To state the result, we call a factorization algebra on $C\times\mathbf{C}^2$ *Dolbeault-conformal* if it is equipped with a quantum current for the centrally extended algebra $\widetilde{\mathscr{E}}(3|6)$. Its coefficient c_{fr} is called the free central charge.

Theorem (Theorem 6.11)**.** *Let $\mathcal{F}$ be a Dolbeault-conformal factorization algebra on $C\times\mathbf{C}^2$ of free central charge c_{fr} (this is the coefficient of the anomaly Θ_{fr}). Deformation by the Euler vector field followed by pushforward,*

$$\mathrm{Loc}_{\pi,\mathsf{Eu}}(\mathcal{F}) = \pi_*\left(\mathcal{F}, \mathrm{d}_{\mathcal{F}} + \mathbf{T}(\mathsf{Eu})\star(-)\right), \tag{1.8}$$

is a chiral conformal factorization algebra on C of central charge

$$c_{2d} = c_{fr}.$$

At the level of symmetry algebras, the deformed exceptional local Lie algebra localizes to holomorphic vector fields on C, together with a locally constant $\mathfrak{sl}(2)$-factor. At the quantum level, Proposition 6.13 identifies the localization of the twisted factorization enveloping algebra with the Virasoro factorization algebra, tensored with the factorization enveloping algebra of that locally constant $\mathfrak{sl}(2)$-symmetry. The six-dimensional anomaly Θ_{fr} maps to the Virasoro cocycle with the same coefficient, which accounts for the equality of central charges. We note that the anomaly associated to the free theory Θ_{fr} is only one of two linearly independent central anomalies for $\mathscr{E}(3|6)_{\mathcal{R}}$. The classification of such anomalies and their role in constructing the holomorphic twist of the higher rank superconformal theories will appear in future work.

For the polarized free tensor multiplet, the localization can be computed directly:

$$\mathrm{Loc}_{\pi,\mathsf{Eu}}(\widetilde{\mathrm{Obs}}) \simeq \mathrm{Bos}, \tag{1.9}$$

where Bos is the chiral-boson factorization algebra on C. This equivalence preserves the conformal structure and has central charge one. When $C = \mathbf{C}$, extraction of a vertex algebra gives the Heisenberg vertex algebra (or chiral boson vertex algebra) of central charge one.

Organization of the paper

Section 2 constructs the holomorphic Lie superalgebra $\mathcal{E}(3|6)_{\mathcal{R}}$, its Dolbeault local Lie algebra $\mathscr{E}(3|6)_{\mathcal{R}}$, and the polarized versions used on product threefolds. It also relates the construction to divergence-free vector fields and closed forms on the Calabi–Yau fivefold $\mathrm{Tot}(\mathcal{R})$.

Section 3 gives the supergeometric interpretation of $E(3|6)$ in terms of Tanaka structures. It identifies the formal automorphisms of the flat model and constructs homogeneous and elliptic examples.

Section 4.4 recalls the holomorphic twist of the free tensor multiplet and constructs the classical L_∞ current.

Section 5 quantizes the free theory and its symmetry. It constructs the quantum current, computes the one-loop anomaly by Riemann–Roch, and describes the resulting currents on a punctured formal three-disk.

Section 6 develops holomorphic localization. It studies the deformation by a vector field on the surface, proves the localization theorem above, computes the localized anomaly and free theory, and extracts the Heisenberg vertex algebra.

Finally, Section A records the parallel localization calculation for holomorphic twists of four-dimensional $\mathcal{N} = 2$ theories.

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Notation and conventions

For a vector bundle E on a manifold X, we write $\mathscr{E}$ for its sheaf of smooth sections and $\mathcal{E}$ for its sheaf of holomorphic sections when E is holomorphic. If E and F are bundles on X and Y, respectively, then

$$E \boxtimes F = p_X^* E \otimes p_Y^* F$$

denotes their external tensor product on $X \times Y$.

Our cochain complexes are cohomologically graded: E^i lies in degree i, and the differential has degree $+1$. The symbol Π denotes reversal of super parity. Cohomological degree and super parity are kept separate unless a consistent $\mathbb{Z}$-grading is explicitly introduced.

A local dg Lie algebra is the sheaf of smooth sections of a graded vector bundle equipped with a differential operator and a bidifferential bracket satisfying the dg Lie identities [CG21]. If $\mathcal{L}$ is a holomorphic Lie algebra or Lie superalgebra, its Dolbeault resolution

$$\Omega^{0,\bullet}(X, \mathcal{L})$$

is a local dg Lie algebra or local dg Lie superalgebra. We use $\mathcal{E}(3|6)_{\mathcal{R}}$ for the holomorphic exceptional Lie superalgebra and $\mathscr{E}(3|6)_{\mathcal{R}}$ for its Dolbeault resolution; tildes denote the consistently graded polarized versions.

2 An exceptional sheaf of Lie superalgebras

In this section we introduce the enhanced exceptional symmetry governing the holomorphic twist of six-dimensional $\mathcal{N} = (2,0)$ superconformal theories. Such twists are defined on any complex threefold equipped with additional geometric data that we describe below. The symmetry takes the form of a *local Lie algebra* on X, in the sense of [CG21, Definition 3.1.3.1]. On $X = \mathbf{C}^3$, polynomial sections give the polynomial form of the exceptional Lie superalgebra, while formal completion at the origin gives the standard linearly compact Lie superalgebra $E(3|6)$ discovered in [Kac98; CK99]. Entire holomorphic sections give the corresponding analytic version.

Following the construction of this exceptional local Lie algebra, we provide an explicit realization of it as a symmetry of the holomorphic twist of the free six-dimensional $\mathcal{N} = (2,0)$ superconformal field theory (otherwise known as the tensor multiplet, the abelian $\mathcal{N} = (2,0)$ theory, or the worldvolume theory of a single fivebrane in M-theory).

2.1 An exceptional holomorphic Lie superalgebra

By a *holomorphic Lie algebra* on a complex manifold X, we mean a holomorphic vector bundle $\mathcal{L}$ equipped with a holomorphic bidifferential operator

$$[-,-]\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L} \tag{2.1}$$

on its sheaf $\mathcal{L}$ of holomorphic sections that makes it into a sheaf of Lie algebras. Similarly, a holomorphic Lie superalgebra is a super vector bundle equipped with a parity-preserving bidifferential operator that satisfies the super Jacobi identity and hence defines a sheaf of Lie superalgebras. Given a holomorphic Lie (super)algebra $\mathcal{L}$, its Dolbeault complex $\Omega^{0,\bullet}(X,\mathcal{L})$ is a local dg Lie (super) algebra in the sense of [CG21].

We turn to the geometric context of this section. For X a complex threefold (a complex manifold of complex dimension three), let $\mathcal{R}$ be a rank-two holomorphic vector bundle on X equipped with a bundle isomorphism

$$\Omega\colon \wedge^2 \mathcal{R} \to \mathcal{K}_X. \tag{2.2}$$

(Thus $\mathcal{R}$ is a "rank-two square root of the canonical bundle.") In what follows, we will work in the context of such a triple $(X,\mathcal{R},\Omega)$; there is a natural site of complex threefolds equipped with such geometric structure that we do not explicitly develop further here. We comment on the relation of these geometric data to the holomorphic twist of $\mathcal{N} = (2,0)$ supersymmetry below.

Let $\mathcal{A}\mathrm{t}(\mathcal{R})$ be the holomorphic Atiyah algebroid of the bundle $\mathcal{R}$ (see [Mac87] for a textbook reference). By definition, its sections are first-order differential operators on sections of $\mathcal{R}$ with central symbol. It fits into the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}\mathrm{nd}(\mathcal{R}) \arrow[r] \arrow[d, "="] & \mathcal{A}\mathrm{t}(\mathcal{R}) \arrow[r, "\#"] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d, "\mathbb{1}"] & 0 \\ 0 \arrow[r] & \mathcal{E}\mathrm{nd}(\mathcal{R}) \arrow[r] & \mathcal{D}\mathrm{iff}^{\leq 1}(\mathcal{R}) \arrow[r, "\mathrm{symb}"] & \mathcal{T}_X \otimes \mathcal{E}\mathrm{nd}(\mathcal{R}) \arrow[r] & 0 \end{tikzcd} \tag{2.3}$$

of sheaves of Lie algebras, where # is the anchor map. Indeed, (2.3) is also a diagram of holomorphic vector bundles on X, but we forewarn the reader that this structure will require some modification in what follows.

Given any $D \in \mathcal{A}\mathrm{t}(\mathcal{R})$, we can define a first-order differential operator acting on sections of $\det(\mathcal{R}) = \wedge^2\mathcal{R}$ by the formula

$$\det(D)(r_1 \wedge r_2) \overset{\text{def}}{=} D(r_1) \wedge r_2 + r_1 \wedge D(r_2). \tag{2.4}$$

This differential operator has the same symbol as D, so it determines a section of $\mathcal{A}\mathrm{t}(\det(\mathcal{R}))$. Recall that we have fixed an isomorphism $\Omega\colon \det(\mathcal{R}) \xrightarrow{\cong} \mathcal{K}_X$, and that any vector field on X canonically determines a first-order differential operator on $\mathcal{K}_X$, the Lie derivative. This leads us to the following definition.

Definition 2.1. The subsheaf $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R}) \subset \mathcal{A}\mathrm{t}(\mathcal{R})$ is defined by the condition

$$\mathcal{A}\mathrm{t}_\Omega(\mathcal{R}) \overset{\text{def}}{=} \{D \in \mathcal{A}\mathrm{t}(\mathcal{R}) \mid \Omega \circ \det(D) \circ \Omega^{-1} = L_{\#D}\}. \tag{2.5}$$

In other words, a first-order differential operator D (with symbol proportional to $\mathbb{1}$) is a section of $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ if, upon using the identification Ω, the operator $\det(D)$ acts by the Lie derivative along its symbol.

It is immediate that $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ is a sheaf of sub Lie algebras. Moreover, the kernel of # restricted to $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ consists of those $\mathcal{O}$-linear operators $A \in \mathcal{E}\mathrm{nd}(\mathcal{R})$ for which the endomorphism of $\det(\mathcal{R})$ induced via (2.4) is zero. But this induced action of A on $\det(\mathcal{R})$ is just multiplication by $\mathrm{tr}(A)$. Therefore we have a short exact sequence

$$0 \longrightarrow \mathfrak{sl}(\mathcal{R}) \longrightarrow \mathcal{A}\mathrm{t}_\Omega(\mathcal{R}) \xrightarrow{\ \#\ } \mathcal{T}_X \longrightarrow 0 \tag{2.6}$$

of sheaves of Lie algebras.

Remark 2.2. If we choose a local frame $\{e_1, e_2\}$ for $\mathcal{R}$, then $\Omega_0 = \Omega(e_1 \wedge e_2)$ becomes a local holomorphic volume form on X, and the condition that $D = (X, A)$ is a section of $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ becomes

$$\mathrm{tr}(A) = \mathrm{div}_{\Omega_0}(X). \tag{2.7}$$

2.1.1. The sequence (2.6) suggests that $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ should admit the structure of a holomorphic Lie algebroid, and indeed it does. However, viewing it as such requires a modification to the $\mathcal{O}_X$-module structure, since the determinant condition (2.5) is not $\mathcal{O}_X$-linear. For f a holomorphic function and $D \in \mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$, the appropriate $\mathcal{O}_X$-module structure is defined by

$$f \cdot D \overset{\text{def}}{=} fD + \frac{1}{2}\#D(f)\mathbb{1}. \tag{2.8}$$

In the first term, we use the standard $\mathcal{O}_X$-module structure on differential operators (left multiplication). Using (2.4), one quickly checks that

$$\det(f \cdot D) = f\det(D) + \#D(f), \tag{2.9}$$

just as is true for the Lie derivative acting on sections of $\mathcal{K}_X$.[1] Thus the condition defining $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ becomes $\mathcal{O}_X$-linear with respect to the corrected $\mathcal{O}_X$-module structure. One further observes that the Lie bracket and anchor from above obey the Lie–Rinehart property with respect to the corrected module structure, so that $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ acquires the structure of a holomorphic Lie algebroid. In particular, it is a holomorphic Lie algebra on X.

2.1.2. We proceed with the construction of the holomorphic Lie superalgebra $\mathcal{E}(3|6)_\mathcal{R}$ which is the main goal of this section. Its even part is what we have just discussed

$$\mathcal{E}(3|6)^0_\mathcal{R} \stackrel{\text{def}}{=} \mathcal{A}\mathrm{t}_\Omega(\mathcal{R}). \tag{2.10}$$

The odd part is the locally free $\mathcal{O}_X$-module

$$\mathcal{E}(3|6)^1_\mathcal{R} \stackrel{\text{def}}{=} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{R}^*. \tag{2.11}$$

In the following paragraphs, we will complete the description of the local Lie superalgebra structure on $\mathcal{E}(3|6)_\mathcal{R}$.

We begin by equipping $\mathcal{E}(3|6)^1_\mathcal{R}$ with the structure of an $\mathcal{E}(3|6)^0_\mathcal{R}$-module. For D a section of $\mathcal{E}(3|6)^0_\mathcal{R}$ and $\alpha = \omega \otimes \rho$ a (decomposable) section of $\mathcal{E}(3|6)^1_\mathcal{R}$, consider the map

$$\mathcal{E}(3|6)^0_\mathcal{R} \to \mathcal{D}\mathrm{iff}^{\leq 1}(\mathcal{E}(3|6)^1_\mathcal{R}), \qquad D \mapsto L_D \tag{2.12}$$

that associates to D the first-order differential operator L_D on sections of $\mathcal{E}(3|6)^1_\mathcal{R}$ defined by the formula

$$L_D(\omega \otimes \rho) \stackrel{\text{def}}{=} L_{\#D}\omega \otimes \rho + \omega \otimes D^*\rho. \tag{2.13}$$

Here, $D^* \in \mathcal{D}\mathrm{iff}^{\leq 1}(\mathcal{R}^*)$ denotes the dual first-order differential operator defined by the formula

$$(D^*\rho)(r) = \#D\,(\rho(r)) - \rho(Dr), \tag{2.14}$$

where r is a section of $\mathcal{R}$. In particular, one has the useful formula

$$(L_D\alpha)(r) = L_{\#D}\alpha(r) - \alpha(Dr), \tag{2.15}$$

which also characterizes L_D.

Lemma 2.3. *For all sections D, D' of $\mathcal{E}(3|6)^0_\mathcal{R}$ one has*

$$L_{[D,D']} = [L_D, L_{D'}] \tag{2.16}$$

as first-order differential operators acting on sections of $\mathcal{E}(3|6)^1_\mathcal{R}$.

Proof. Let α be a section of $\mathcal{E}(3|6)^1_\mathcal{R} = \Omega^1 \otimes_\mathcal{O} \mathcal{R}^*$, and r a section of $\mathcal{R}$. Then

$$L_D(L_{D'}\alpha)(r) = L_{\#D}L_{\#D'}\alpha(r) - L_{\#D}\alpha(D'r) - L_{\#D'}\alpha(Dr) + \alpha(D'Dr), \tag{2.17}$$

[1] More generally, if R is rank r then this compatible $\mathcal{O}_X$-module structure is $fD + \frac{1}{r}\#Df\mathbb{1}$. We remark that the formula defines a one-parameter family of $\mathcal{O}_X$-module structures on the Atiyah algebroid, each of which fits into a Lie algebroid structure with the same bracket and anchor.

from which we see that Taking the supercommutator and cancelling the mixed terms gives

$$[L_D, L_{D'}](\alpha)(r) = [L_{\#D}, L_{\#D'}]\alpha(r) - \alpha([D, D']r). \tag{2.18}$$

Since the anchor and the Lie derivative preserve brackets, the right-hand side is $L_{[D,D']}\alpha(r)$. □

We will also need the following easy lemma.

Lemma 2.4. *Let α, β be sections of $\mathcal{E}(3|6)^1_{\mathcal{R}}$. Then there exists a unique vector field $\mu_{\alpha,\beta} = \mu_{\beta,\alpha}$ on X such that*

$$\iota_{\mu_{\alpha,\beta}}\Omega(r \wedge s) = \alpha(r) \wedge \beta(s) + \beta(r) \wedge \alpha(s) \tag{2.19}$$

for arbitrary sections r, s of $\mathcal{R}$.

Proof. Given the sections α and β, the formula

$$(r, s) \mapsto \alpha(r) \wedge \beta(s) + \beta(r) \wedge \alpha(s) \tag{2.20}$$

is clearly alternating and $\mathcal{O}_X$-bilinear. Thus it determines a bundle map from $\wedge^2\mathcal{R}$ to Ω^2_X; this is equivalently a section of $\wedge^2\mathcal{R}^* \otimes \Omega^2_X$, which is in turn isomorphic to $\mathcal{T}_X$ via the (inverse dual of the) bundle isomorphism Ω. □

To a pair of sections α, β of $\mathcal{E}(3|6)^1_{\mathcal{R}}$ as above, we can symmetrically associate a differential operator $D_{\alpha,\beta} = D_{\beta,\alpha} \in \mathcal{D}\mathrm{iff}^{\leq 1}(\mathcal{R})$ by requiring that

$$\Omega(D_{\alpha,\beta}(r) \wedge s) = \mathrm{d}\alpha(r) \wedge \beta(s) + \mathrm{d}\beta(r) \wedge \alpha(s) \tag{2.21}$$

for all sections r, s of $\mathcal{R}$.

Proposition 2.5. *For any sections α, β, γ of $\mathcal{E}(3|6)^1_{\mathcal{R}}$, the following are true concerning the differential operator $D_{\alpha,\beta}$:*

(a) *$D_{\alpha,\beta}$ is a section of $\mathcal{E}(3|6)^0_{\mathcal{R}}$.*

(b) *For D a section of $\mathcal{E}(3|6)^0_{\mathcal{R}}$, one has*

$$[D, D_{\alpha,\beta}] = D_{L_D\alpha,\beta} + D_{\alpha,L_D\beta}. \tag{2.22}$$

In other words, the assignment $\alpha, \beta \mapsto D_{\alpha,\beta}$ is $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$-equivariant.

(c) *For all sections α, β, γ of $\mathcal{E}(3|6)^1_{\mathcal{R}}$ one has:*

$$L_{D_{\alpha,\beta}}\gamma + \textit{cyclic} = 0. \tag{2.23}$$

Proof. We begin with (a). For $f \in \mathcal{O}_X$ and r, s sections of $\mathcal{R}$ one has

$$\begin{aligned}\Omega(D_{\alpha,\beta}(fr) \wedge s) &= f\Omega(D_{\alpha,\beta}r \wedge s) + \mathrm{d}f \wedge \big(\alpha(r) \wedge \beta(s) - \alpha(s) \wedge \beta(r)\big)\\ &= f\Omega(D_{\alpha,\beta}r \wedge s) + \mathrm{d}f \wedge \iota_{\mu_{\alpha,\beta}}\Omega(r \wedge s),\end{aligned} \tag{2.24}$$

the latter equality by definition of $\mu_{\alpha,\beta}$ (see lemma 2.4). Since $\mathrm{d}f \wedge \iota_\mu \Omega = \mu(f)\Omega$ for any section Ω of $\mathcal{K}_X$, this can be written as

$$\Omega(D_{\alpha,\beta}(fr) \wedge s) = f\Omega(D_{\alpha,\beta} r \wedge s) + \mu_{\alpha,\beta}(f)\Omega(r \wedge s). \tag{2.25}$$

Since Ω is an isomorphism and s is arbitrary, we conclude that

$$D_{\alpha,\beta}(fr) = fD_{\alpha,\beta} r + \mu_{\alpha,\beta}(f)r. \tag{2.26}$$

Thus $D_{\alpha,\beta}$ is a first-order differential operator with scalar principal symbol, and so a section of $\mathcal{A}\mathrm{t}(\mathcal{R})$. Its symbol is precisely the vector field $\mu_{\alpha,\beta}$.

We now check the determinant condition. Antisymmetrizing (2.21) in r and s, we see that

$$\begin{aligned}\Omega(D_{\alpha,\beta} r \wedge s) + \Omega(r \wedge D_{\alpha,\beta} s) = \\ d\alpha(r) \wedge \beta(s) + d\beta(r) \wedge \alpha(s) - d\alpha(s) \wedge \beta(r) - d\beta(s) \wedge \alpha(r).\end{aligned} \tag{2.27}$$

But the right hand side is a total derivative, which we rewrite as $\mathrm{d}\iota_{\mu_{\alpha,\beta}}\Omega(r \wedge s)$ recalling lemma 2.4. Since $\Omega(r \wedge s)$ is a top form, we conclude that

$$\Omega(D_{\alpha,\beta} r \wedge s) + \Omega(r \wedge D_{\alpha,\beta} s) = L_{\#D_{\alpha,\beta}}\Omega(r \wedge s). \tag{2.28}$$

This is precisely the determinant condition, so that $D_{\alpha,\beta}$ is a section of $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$. This proves (a).

For (b) we will prove equation (2.22) by pairing both sides with $\Omega(- \wedge s)$. Applying the determinant condition to D, we have

$$\Omega([D, D_{\alpha,\beta}]r \wedge s) = L_{\#D}\Omega(D_{\alpha,\beta} r \wedge s) - \Omega(D_{\alpha,\beta} r \wedge Ds) - \Omega(D_{\alpha,\beta}(Dr) \wedge s) \tag{2.29}$$

Using (2.21), the Leibniz rule, and the fact that $[L_\mu, \mathrm{d}] = 0$, we can write the first term as

$$\begin{aligned}L_{\#D}\Omega(D_{\alpha,\beta} r \wedge s) = (\mathrm{d}L_{\#D}\alpha(r)) \wedge \beta(s) + \mathrm{d}\alpha(r) \wedge L_{\#D}\beta(s) \\ + (\mathrm{d}L_{\#D}\beta(r)) \wedge \alpha(s) + \mathrm{d}\beta(r) \wedge L_{\#D}\alpha(s).\end{aligned} \tag{2.30}$$

Thus equation (2.29) becomes

$$\begin{aligned}\Omega([D, D_{\alpha,\beta}]r \wedge s) = \mathrm{d}((L_D\alpha)(r)) \wedge \beta(s) + \mathrm{d}(\beta(r)) \wedge (L_D\alpha)(s) \\ + \mathrm{d}(\alpha(r)) \wedge (L_D\beta)(s) + \mathrm{d}((L_D\beta)(r)) \wedge \alpha(s).\end{aligned} \tag{2.31}$$

Rearranging, we arrive at

$$\Omega([D, D_{\alpha,\beta}]r \wedge s) = \Omega((D_{L_D\alpha,\beta} + D_{\alpha,L_D\beta})r \wedge s) \tag{2.32}$$

Item (b) follows.

Finally, we prove (c). We must show that

$$L_{D_{\alpha,\beta}}\gamma + L_{D_{\beta,\gamma}}\alpha + L_{D_{\gamma,\alpha}}\beta = 0.$$

It is enough to evaluate this expression on an arbitrary local section r of $\mathcal{R}$. We write $\mu_{\alpha,\beta} = \#D_{\alpha,\beta}$ as before. By (2.15), or by definition, we have

$$(L_{D_{\alpha,\beta}}\gamma)(r) = L_{\mu_{\alpha,\beta}}(\gamma(r)) - \gamma(D_{\alpha,\beta}r).$$

Applying Cartan's magic formula, the cyclic sum becomes

$$\sum_{\text{cyc}}(L_{D_{\alpha,\beta}}\gamma)(r) = \sum_{\text{cyc}}\Big(\iota_{\mu_{\alpha,\beta}}\mathrm{d}(\gamma(r)) - \gamma(D_{\alpha,\beta}r)\Big) + \mathrm{d}\sum_{\text{cyc}}\iota_{\mu_{\alpha,\beta}}\gamma(r). \tag{2.33}$$

We will show that both terms vanish independently, starting with the second term. The claim is local in α, β, γ, so it suffices to check it for decomposable sections

$$\alpha = \omega \otimes \tau, \qquad \beta = \eta \otimes \nu, \qquad \gamma = \theta \otimes \lambda,$$

where ω, η, θ are one-forms and τ, ν, λ are sections of $\mathcal{R}^*$.

Choose a local frame e_1, e_2 of $\mathcal{R}$, so that $\Omega_0 = \Omega(e_1 \wedge e_2)$ is a locally defined holomorphic volume form. Let $\epsilon \in \wedge^2\mathcal{R}^*$ be such that $\Omega(u \wedge v) = \epsilon(u, v)\Omega_0$ for any sections u, v of $\mathcal{R}$. For $\rho, \sigma \in \mathcal{R}^*$, define the locally defined function $[\rho, \sigma]$ by the formula

$$\rho \wedge \sigma = [\rho, \sigma]\epsilon.$$

Then the defining property of $\mu_{\alpha,\beta}$ gives $\iota_{\mu_{\alpha,\beta}}\Omega_0 = [\tau, \nu]\,\omega \wedge \eta$.

Now we consider the term $\iota_{\mu_{\alpha,\beta}}\gamma(r)$ appearing in the cyclic sum. Multiplying by Ω_0 and applying the above, we get

$$\iota_{\mu_{\alpha,\beta}}\gamma(r)\,\Omega_0 = \lambda(r)[\tau, \nu]\,\theta \wedge \omega \wedge \eta.$$

Taking the cyclic sum gives

$$\begin{aligned}&\Big(\iota_{\mu_{\alpha,\beta}}\gamma(r) + \iota_{\mu_{\beta,\gamma}}\alpha(r) + \iota_{\mu_{\gamma,\alpha}}\beta(r)\Big)\,\Omega_0 \\ &\qquad = \omega \wedge \eta \wedge \theta\;\left(\lambda(r)[\tau, \nu] + \tau(r)[\nu, \lambda] + \nu(r)[\lambda, \tau]\right).\end{aligned}$$

The coefficient in parentheses is the contraction of $\tau \wedge \nu \wedge \lambda$ with r. Since $\mathcal{R}$ has rank two, this coefficient vanishes. Thus the last term in (2.33) vanishes.

It remains to show that

$$\sum_{\text{cyc}}\Big(\iota_{\mu_{\alpha,\beta}}\mathrm{d}(\gamma(r)) - \gamma(D_{\alpha,\beta}r)\Big) = 0. \tag{2.34}$$

Again, this is local, so we use the same decomposable notation. Set $f = \tau(r)$, $g = \nu(r)$, $h = \lambda(r)$, and $a = [\tau, \nu]$, $b = [\nu, \lambda]$, $c = [\lambda, \tau]$. Thus

$$\iota_{\mu_{\alpha,\beta}}\Omega = a\,\omega \wedge \eta, \qquad \iota_{\mu_{\beta,\gamma}}\Omega = b\,\eta \wedge \theta, \qquad \iota_{\mu_{\gamma,\alpha}}\Omega = c\,\theta \wedge \omega.$$

For a two-form ρ and a one-form κ, write $[\rho, \kappa]_\Omega$ for the unique function satisfying $\rho \wedge \kappa = [\rho, \kappa]_\Omega\Omega_0$.

Since $\gamma(r) = h\theta$, the first term is $\iota_{\mu_{\alpha,\beta}} \mathrm{d}(h\theta)$. For the second term, the defining formula for $D_{\alpha,\beta}$ gives

$$\Omega(D_{\alpha,\beta} r \wedge s) = \mathrm{d}(f\omega) \wedge \nu(s)\eta + \mathrm{d}(g\eta) \wedge \tau(s)\omega.$$

Equivalently,

$$\epsilon(D_{\alpha,\beta} r, s)\Omega_0 = \nu(s)\mathrm{d}(f\omega) \wedge \eta + \tau(s)\mathrm{d}(g\eta) \wedge \omega.$$

Pairing $D_{\alpha,\beta} r$ with λ, and using $[\lambda, \nu] = -[\nu, \lambda] = -b$ and $[\lambda, \tau] = c$, we obtain

$$\lambda(D_{\alpha,\beta} r) = -b[\mathrm{d}(f\omega), \eta]_\Omega + c[\mathrm{d}(g\eta), \omega]_\Omega.$$

Therefore, if we set

$$S_{\alpha,\beta,\gamma}(r) \overset{\text{def}}{=} \iota_{\mu_{\alpha,\beta}} \mathrm{d}(\gamma(r)) - \gamma(D_{\alpha,\beta} r) \tag{2.35}$$

then we have the identity

$$S_{\alpha,\beta,\gamma}(r) = \iota_{\mu_{\alpha,\beta}} \mathrm{d}(h\theta) + b\,\theta[\mathrm{d}(f\omega), \eta]_\Omega - c\,\theta[\mathrm{d}(g\eta), \omega]_\Omega. \tag{2.36}$$

Similarly,

$$S_{\beta,\gamma,\alpha}(r) = \iota_{\mu_{\beta,\gamma}} \mathrm{d}(f\omega) + c\,\omega[\mathrm{d}(g\eta), \theta]_\Omega - a\,\omega[\mathrm{d}(h\theta), \eta]_\Omega, \tag{2.37}$$

and

$$S_{\gamma,\alpha,\beta}(r) = \iota_{\mu_{\gamma,\alpha}} \mathrm{d}(g\eta) + a\,\eta[\mathrm{d}(h\theta), \omega]_\Omega - b\,\eta[\mathrm{d}(f\omega), \theta]_\Omega. \tag{2.38}$$

We now add (2.36), (2.37), and (2.38). We use the following elementary identity: if μ is the unique vector field such that $\iota_\mu \Omega = q\,\kappa_1 \wedge \kappa_2$, then for every two-form ρ,

$$\iota_\mu \rho = -q\,\kappa_2[\rho, \kappa_1]_\Omega + q\,\kappa_1[\rho, \kappa_2]_\Omega. \tag{2.39}$$

Indeed, both sides are one-forms, and after wedging with an arbitrary one-form χ, both give $\rho \wedge \chi \wedge \iota_\mu \Omega / \Omega$.

Apply (2.39) to $\rho = \mathrm{d}(f\omega)$ and to the vector field $\mu_{\beta,\gamma}$, for which $\iota_{\mu_{\beta,\gamma}} \Omega = b\,\eta \wedge \theta$. We get

$$\iota_{\mu_{\beta,\gamma}} \mathrm{d}(f\omega) = -b\,\theta[\mathrm{d}(f\omega), \eta]_\Omega + b\,\eta[\mathrm{d}(f\omega), \theta]_\Omega.$$

These two terms cancel exactly with the two terms involving $\mathrm{d}(f\omega)$ in (2.36) and (2.38).

Similarly, applying (2.39) to $\rho = \mathrm{d}(g\eta)$ and $\mu_{\gamma,\alpha}$, where $\iota_{\mu_{\gamma,\alpha}} \Omega = c\,\theta \wedge \omega$, gives

$$\iota_{\mu_{\gamma,\alpha}} \mathrm{d}(g\eta) = -c\,\omega[\mathrm{d}(g\eta), \theta]_\Omega + c\,\theta[\mathrm{d}(g\eta), \omega]_\Omega.$$

These cancel exactly with the terms involving $\mathrm{d}(g\eta)$.

Finally, applying (2.39) to $\rho = \mathrm{d}(h\theta)$ and $\mu_{\alpha,\beta}$, where $\iota_{\mu_{\alpha,\beta}} \Omega = a\,\omega \wedge \eta$, gives

$$\iota_{\mu_{\alpha,\beta}} \mathrm{d}(h\theta) = -a\,\eta[\mathrm{d}(h\theta), \omega]_\Omega + a\,\omega[\mathrm{d}(h\theta), \eta]_\Omega.$$

These cancel exactly with the terms involving $\mathrm{d}(h\theta)$. Hence

$$S_{\alpha,\beta,\gamma}(r) + S_{\beta,\gamma,\alpha}(r) + S_{\gamma,\alpha,\beta}(r) = 0,$$

which proves (2.34). This proves (c). □

2.1.3. Proposition 2.5, together with Lemma 2.3, brings us to the main definition of this section.

Definition 2.6. Let X be a complex threefold, $R \to X$ a rank two holomorphic vector bundle, and $\Omega\colon \wedge^2 R \to \mathcal{K}_X$ an isomorphism. Define the super $\mathcal{O}_X$-module

$$\mathcal{E}(3|6)_{\mathcal{R}} \overset{\text{def}}{=} \mathcal{A}\mathrm{t}_{\Omega}(\mathcal{R}) \oplus \left(\Omega^1_X \otimes_{\mathcal{O}_X} \Pi\mathcal{R}^*\right). \tag{2.40}$$

Then the following formulas endow $\mathcal{E}(3|6)_{\mathcal{R}}$ with the structure of a holomorphic Lie superalgebra on X:

$$\begin{aligned} [D, D'] &= \text{commutator of differential operators;} \\ [D, \alpha] &= -[\alpha, D] = L_D \alpha; \\ [\alpha, \beta] &= D_{\alpha,\beta}. \end{aligned} \tag{2.41}$$

Here D, D' denote even sections and α, β denote odd sections.

2.1.4. Polarized version. Suppose that $\mathfrak{g} = \mathfrak{g}^{\underline{0}} \oplus \Pi\mathfrak{g}^{\underline{1}}$ is a Lie superalgebra. A *consistent* $\mathbf{Z}$-grading on $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \oplus_{i\in\mathbf{Z}}\mathfrak{g}^i$ with the property that

$$\mathfrak{g}^{\underline{0}} = \bigoplus_i \mathfrak{g}^{2i}, \qquad \mathfrak{g}^{\underline{1}} = \bigoplus_i \mathfrak{g}^{2i+1}, \tag{2.42}$$

and such that the bracket is degree zero for this grading.

Given a consistent $\mathbf{Z}$-grading on a Lie superalgebra $\mathfrak{g}$ we let $\widetilde{\mathfrak{g}}$ be the $\mathbf{Z}$-graded Lie algebra whose degree i part is $\mathfrak{g}^i$. In other words, $\widetilde{\mathfrak{g}} = \oplus_{i\in\mathbf{Z}}\mathfrak{g}^i[-i]$.

For any $\mathcal{R}$ with $\wedge^2\mathcal{R} \cong K_X$ we obtain a holomorphic Lie *super*algebra $\mathcal{E}(3|6)_{\mathcal{R}}$ on X. We point out a variant of this definition: If $\mathcal{R}$ splits as the sum of two line bundles, this Lie superalgebra admits a consistent $\mathbf{Z}$-grading, hence leads to a $\mathbf{Z}$-graded holomorphic Lie algebra on X. We obtain this grading by placing one of the lines in degree $+1$ and the other in degree -1.

For threefolds X that are a product of a complex curve and a complex surface, a natural choice of such a pair of line bundles is available. Indeed, suppose C is a complex curve and S a complex surface. Define the holomorphic vector bundle

$$\mathcal{R} = \pi_C^*\mathcal{K}_C \oplus \pi_S^*\mathcal{K}_S \tag{2.43}$$

on $X = C \times S$. Let Ω be the apparent isomorphism $\wedge^2\mathcal{R} \cong \mathcal{K}_C \boxtimes \mathcal{K}_S = \mathcal{K}_{C\times S}$.

Lemma 2.7. *Let $\mathcal{R}$ be as in (2.43). Then $\mathcal{E}(3|6)_{\mathcal{R}}$ admits a consistent $\mathbf{Z}$-grading.*

Proof. With $\mathcal{R}$ as in (2.43), $\mathcal{A}\mathrm{t}_{\Omega}(\mathcal{R})$ acquires a grading by thinking of the integer grading on $\mathcal{R}$ as defining a $\mathfrak{gl}(1)$ action, which then induces an action on differential operators. Explicitly, we see that $\mathfrak{sl}(\mathcal{R})$ decomposes as

$$\mathfrak{sl}(\mathcal{R}) = \mathcal{T}_C \boxtimes \mathcal{K}_S \oplus \mathcal{O}_{C\times S} \oplus \mathcal{K}_C \boxtimes \wedge^2\mathcal{T}_S, \tag{2.44}$$

with the summands in degrees -2, 0, and 2 respectively,

The grading on $\mathcal{R}$ induces a grading on $\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{R}^*$ in obvious fashion. One checks immediately from the form of the brackets (2.41) that this is a consistent $\mathbf{Z}$-grading. $\square$

From this lemma, we obtain a holomorphic $\mathbf{Z}$-graded Lie algebra on $X = C \times S$ that we denote $\widetilde{\mathcal{E}}(3|6)$. From the proof of the lemma we see that $\widetilde{\mathcal{E}}(3|6)$ is concentrated in cohomological degrees $-2, \ldots, 2$.

The degree 1 part of $\widetilde{\mathcal{E}}(3|6)$ is

$$\mathcal{O}_C \boxtimes \mathcal{T}_S \oplus \Omega^1_C \boxtimes \wedge^2 \mathcal{T}_S. \tag{2.45}$$

2.1.5. A *local* dg Lie algebra $\mathscr{L}$ on a smooth manifold M, in the sense of [CG21], is the sheaf of *smooth* sections of a $\mathbf{Z}$-graded smooth vector bundle equipped with a differential operator

$$\mathrm{d} \colon \mathscr{L} \to \mathscr{L}[1] \tag{2.46}$$

of degree one and a bidifferential operator

$$[-,-] \colon \mathscr{L} \times \mathscr{L} \to \mathscr{L} \tag{2.47}$$

which turn $\mathscr{L}$ into a sheaf of Lie algebras on M. Given any holomorphic Lie algebra $\mathcal{L}$ we obtain a local dg Lie algebra through its Dolbeault complex

$$\mathscr{L} = \Omega^{0,\bullet}(X, \mathcal{L})$$

where $\mathrm{d} = \bar\partial$ is the $\bar\partial$-operator underlying $\mathcal{L}$ and $[-,-]_{\mathcal{L}}$ is extended to a bracket on $\mathscr{L}$ using the wedge product of Dolbeault forms. The analogous construction can be made for local dg Lie *super*algebras.

Applied to our situation, we see that

$$\mathscr{E}(3|6)_{\mathcal{R}} \overset{\text{def}}{=} \Omega^{0,\bullet}(X, \mathcal{E}(3|6)_{\mathcal{R}}) \tag{2.48}$$

is naturally a local dg Lie superalgebra on X.

Similarly, when $X = C \times S$ we apply the Dolbeault complex to the $\mathbf{Z}$-graded holomorphic Lie algebra $\widetilde{\mathcal{E}}(3|6)$ (and then totalize the two $\mathbf{Z}$-gradings) introduced in 2.1.4 to obtain the local dg Lie algebra

$$\widetilde{\mathscr{E}}(3|6) \overset{\text{def}}{=} \Omega^{0,\bullet}(C \times S, \widetilde{\mathcal{E}}(3|6)). \tag{2.49}$$

2.2 The relationship to Calabi–Yau fivefolds

In this section, we quickly sketch the relationship of our geometric site to the geometry of complex three-dimensional submanifolds of Calabi–Yau fivefolds. This indicates how our constructions connect to fivebranes in the holomorphic twist of eleven-dimensional supergravity [RSW23].

Let $(X, \mathcal{R}, \Omega)$ be as in the previous section. Then we can form the local fivefold $\mathrm{Tot}(\mathcal{R})$. We let $E_{\mathcal{R}}$ denote the fiberwise Euler vector field.

There is a natural sequence

$$0 \to \pi^* R \to T\,\mathrm{Tot}(\mathcal{R}) \to \pi^* TX \to 0 \tag{2.50}$$

of vector bundles on $\mathrm{Tot}(\mathcal{R})$. From this, we deduce that

$$K_{\mathrm{Tot}(\mathcal{R})} = \pi^* \left(\mathcal{K}_X \otimes \det(\mathcal{R})^{-1}\right), \tag{2.51}$$

so that the isomorphism $\Omega\colon \wedge^2 \mathcal{R} \to \mathcal{K}_X$ gives $\mathrm{Tot}(\mathcal{R})$ the structure of a local Calabi–Yau fivefold. Thus we have a notion of divergence-free vector field on $\mathrm{Tot}(\mathcal{R})$; we will denote the sheaf of these by $\mathrm{Vect}_\Omega(\mathrm{Tot}(\mathcal{R}))$. We note further that the Calabi–Yau form Ω has homogeneous $E_{\mathcal{R}}$-weight $+2$.

We have the diagram of injections of sheaves of Lie algebras

$$\begin{tikzcd}
\mathcal{V}\mathrm{ect}_\Omega(\mathrm{Tot}(\mathcal{R}))^{(0)} \arrow[r] \arrow[d] & \mathcal{V}\mathrm{ect}_\Omega(\mathrm{Tot}(\mathcal{R})) \arrow[d] \\
\mathcal{V}\mathrm{ect}(\mathrm{Tot}(\mathcal{R}))^{(0)} \arrow[r] & \mathcal{V}\mathrm{ect}(\mathrm{Tot}(\mathcal{R})),
\end{tikzcd} \tag{2.52}$$

where the subsheaves in the left-hand column is defined by the condition of having E_R-weight zero, and the subsheaves at top are defined by the condition of divergence-freeness. We note that $\pi_* \mathcal{V}\mathrm{ect}(\mathrm{Tot}(\mathcal{R}))^{(0)}$ is precisely the sheaf $\mathcal{A}\mathrm{t}(\mathcal{R})$ on X consisting of sections of the Atiyah algebroid; indeed, this is one of its standard descriptions. Furthermore, one identifies $\pi_* \mathcal{V}\mathrm{ect}(\mathrm{Tot}(\mathcal{R}))^{(0)} = \mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$.

Proposition 2.8. *The pushforward $\pi_* \mathrm{Vect}_\Omega(\mathrm{Tot}(\mathcal{R}))^{(0)}$ of the sheaf of weight-zero divergence-free vector fields on $\mathrm{Tot}(\mathcal{R})$ along the projection to X defines a locally free sheaf of Lie–Rinehart algebras, and thus a holomorphic Lie algebroid, on X.*

Proof. This is essentially immediate via the identification with $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$, but we sketch the proof for convenience. Consider the subsequence of (2.50) of $E_{\mathcal{R}}$-weight zero. The divergence-free weight-zero vector fields form a subsheaf of the sheaf of sections of the middle term, and $E_{\mathcal{R}}$ itself defines a globally nonvanishing section of the first term.

For sections χ of $\pi_* \mathrm{Vect}_\Omega(\mathrm{Tot}(\mathcal{R}))^{(0)}$, the formula

$$f \cdot \chi \coloneqq f\chi - \frac{1}{2}\chi(f)E_R, \tag{2.53}$$

where we implicitly view a function on X as a weight-zero function on $\mathrm{Tot}(\mathcal{R})$, defines an $\mathcal{O}_X$-module structure. To see this, one notes that

$$f \cdot (g \cdot \chi) = f \cdot \left(g\chi - \frac{1}{2}\chi(g)E_R\right) = fg\chi - \frac{1}{2}f\chi(g)E_R - \frac{1}{2}g\chi(f)E_R = (fg)\cdot\chi,$$

since $E_R(f) = 0$. Furthermore, the right-hand side of (2.53) is divergence-free, since $\mathrm{div}(E_{\mathcal{R}}) = 2$ and $\chi(f)$ has weight zero.

Since E_R is vertical, the natural map $D\pi : T\,\mathrm{Tot}(\mathcal{R}) \to \pi^*TX$ defines a map of C^∞_X-modules (and Lie algebras) from $\pi_* \mathrm{Vect}_\Omega(\mathrm{Tot}(\mathcal{R}))^{(0)}$ to $\mathrm{Vect}(X)$. Taking it to define the anchor, we obtain a sheaf of Lie–Rinehart algebras; one observes that

$$\begin{aligned}[\chi', f\cdot\chi] &= [\chi', f\chi] - \frac{1}{2}[\chi', \chi(f)E_R] \\ &= f[\chi',\chi] + \chi'(f)\chi - \frac{1}{2}\chi'(\chi(f))E_R \\ &= f[\chi',\chi] - \frac{1}{2}[\chi',\chi](f)E_R + \chi'(f)\chi - \frac{1}{2}\chi(\chi'(f))E_R \\ &= f\cdot[\chi',\chi] + \chi'(f)\cdot\chi,\end{aligned}$$

which is the last of the Lie–Rinehart properties. □

Consider now the sheaf of holomorphic closed two-forms on $\mathrm{Tot}(\mathcal{R})$ of E_R-weight one, which we denote $\Omega^{2,\mathrm{cl}}(\mathrm{Tot}(\mathcal{R}))^{(1)}$.

Proposition 2.9. *There is an isomorphism*

$$\Omega^1_X \otimes \mathcal{R}^* \xrightarrow{\cong} \pi_*\Omega^{2,\mathrm{cl}}(\mathrm{Tot}(\mathcal{R}))^{(1)}$$

of sheaves of locally free $\mathcal{O}_X$-modules on X.

Proof. Two-forms of weight one on $\mathrm{Tot}(\mathcal{R})$ can be identified with sections of the bundle $(R^*\otimes\wedge^2T^*X)\oplus(T^*X\wedge R^*)$. Working locally, a section of this bundle takes the form

$$w_i\beta_i + dw_i\wedge\alpha_i,$$

where w_i is a local basis of sections of R^* (linear fiberwise coordinates on $\mathrm{Tot}(\mathcal{R})$), β_i are two-forms on X, and α_i are one-forms on X. Demanding that such an element is closed means that $\beta_i = d\alpha_i$. □

Armed with these two sheaves, we can make the following observation, whose proof is straightforward.

Proposition 2.10. *On any threefold $(X,\mathcal{R},\Omega)$ equipped with geometric structure as above, the sheaf*

$$\pi_*\left[\mathcal{V}\mathrm{ect}_\Omega(\mathrm{Tot}(\mathcal{R}))^{(0)}\oplus\Pi\Omega^{2,\mathrm{cl}}(\mathrm{Tot}(\mathcal{R}))^{(1)}\right] \tag{2.54}$$

admits the structure of a holomorphic super Lie algebra on X, with bracket defined as follows:

- *The bracket of even elements is the Lie bracket of divergence-free vector fields of weight zero on* $\mathrm{Tot}(\mathcal{R})$.
- *The action of even elements on odd elements is the Lie derivative of closed two-forms of weight one along divergence-free vector fields of weight zero.*
- *The bracket of odd elements is the wedge product, followed by the use of Ω to identify closed four-forms of weight two on* $\mathrm{Tot}(\mathcal{R})$ *with divergence-free vector fields of weight zero.*

The result is equivalent to the holomorphic super Lie algebra $\mathcal{E}(3|6)_\mathcal{R}$ from Definition 2.6.

Remark 2.11. In [RSW23], we remarked on the relation of holomorphic eleven-dimensional supergravity to the exceptional super Lie algebra $E(5|10)$. Kac observes the existence of a consistent $\mathbf{Z}$-grading on $E(5|10)$ whose weight-zero subalgebra is precisely $E(3|6)$. Our observations here recover Kac's construction in the case $\mathbf{C}^3 \subset \mathbf{C}^5$, and we see them as a globalization of his results in a geometric context arising in M-theory.

3 Tanaka structures and exceptional geometry

In this section we present $E(3|6)$ as the infinitesimal automorphisms of a precise supergeometric structure. Recall that a super Riemann surface is a complex supermanifold of dimension $1|1$ equipped with a rank-$0|1$ distribution $\mathscr{D} \subset \mathrm{T}_M$ whose Levi bracket is an isomorphism

$$\mathscr{D}^{\otimes 2} \xrightarrow{\cong} \mathrm{T}_M/\mathscr{D}.$$

The distribution is therefore maximally nonintegrable. Its symbol algebra is locally constant, and the infinitesimal automorphisms of its flat model form the superconformal algebra. This familiar example is the simplest instance of a *Tanaka structure*: the filtration generated by $\mathscr{D}$, its graded Levi bracket, and the allowed changes of adapted frame together encode the superconformal geometry.

The $E(3|6)$-structures considered here are an exceptional higher-dimensional analogue. The odd line of a super Riemann surface is replaced by a rank-$0|6$ distribution on a $3|6$-dimensional complex supermanifold. Its Levi bracket has a fixed two-step nilpotent symbol $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$, and a reduction to the group G_0 records the compatible degree-zero tensorial data. The Tanaka prolongation of this symbol together with its G_0-action is $E(3|6)$, just as the prolongation of the super Riemann surface symbol recovers its local superconformal symmetries.

We first review the required Tanaka terminology, then define $E(3|6)$-structures and study their flat and homogeneous models. This geometric reformulation complements the exceptional sheaf of Lie superalgebras constructed in Section 2, but is not needed in the later field-theoretic arguments.

3.1 Prolongation in context

We briefly recall the role of prolongation in geometry and explain why the particular prolongation used below appears in the holomorphic twist of a six-dimensional superconformal theory. For an ordinary flat G_0-structure, one begins with the translations V and the infinitesimal structure group $\mathfrak{g}_0 \subseteq \mathfrak{gl}(V)$. Its algebra of infinitesimal automorphisms can contain positive-degree polynomial vector fields in addition to the affine algebra $V \rtimes \mathfrak{g}_0$. These higher terms are determined recursively by the requirement that they preserve the flat structure; this is the classical prolongation construction [GS64]. For example, the prolongation of an orthogonal structure stops at the Euclidean or Poincaré algebra, whereas the conformal structure in dimension at least three acquires the special conformal transformations. In complex dimension one the conformal prolongation is infinite-dimensional and gives the formal holomorphic vector fields.

Tanaka's construction is the nonintegrable analogue of this procedure. The abelian translation algebra V is replaced by a fundamental negatively graded nilpotent Lie algebra $\mathfrak{m} = \bigoplus_{j<0} \mathfrak{m}_j$, called the *symbol*, and $\mathfrak{g}_0$ acts on it by grading-preserving derivations. The maximal transitive graded Lie algebra with prescribed nonpositive part $\mathfrak{m} \rtimes \mathfrak{g}_0$ is its Tanaka prolongation. In supergeometry the symbol is often a supertranslation algebra. Altomani and Santi classified the maximal transitive prolongations of the super-Poincaré algebras [AS14]; in the dimensions in which finite-dimensional superconformal symmetry exists, their results recover the familiar algebras in Nahm's classification [Nah78; Shn88]. Thus the passage from a supertranslation algebra to a larger superconformal algebra is itself a Tanaka-prolongation phenomenon.

Infinite-dimensional linearly compact Lie superalgebras provide a second, exceptional manifestation of the same construction. Kac's classification shows that, besides the Cartan-type series and their filtered deformations, there are exactly five exceptional simple linearly compact Lie superalgebras [Kac98, Theorem 6.3]:

$$E(1|6), \qquad E(3|6), \qquad E(3|8), \qquad E(4|4), \qquad E(5|10).$$

The notation records the dimension of a supermanifold on which each algebra has a transitive formal realization. Cheng and Kac exhibited a consistent grading of $E(3|6)$ whose nonpositive part is

$$\mathfrak{m}_{-2} = A, \qquad \mathfrak{m}_{-1} = \Pi(A^* \otimes B^*), \qquad \mathfrak{g}_0 = \mathfrak{sl}(A) \oplus \mathfrak{sl}(B) \oplus \mathbb{C}, \tag{3.1}$$

where $\dim A = 3$ and $\dim B = 2$, and identified its maximal transitive prolongation with $E(3|6)$ [CK99, case (C4), §4.4]. The definitions below make the bracket on $\mathfrak{m}$ precise, and Theorem 3.9 realizes this algebra as the formal infinitesimal automorphisms of the corresponding flat Tanaka structure.

The classification result alone does not explain why this exceptional prolongation should govern a holomorphically twisted six-dimensional theory. That connection is supplied by the derived local formulation of superconformal symmetry developed in [Hah+24]. We record the result that motivates the geometry of this paper.

Theorem 3.1 ([Hah+24]). *Let Q^{hol} be a holomorphic supercharge in the six-dimensional $\mathcal{N} = (2,0)$ supersymmetry algebra. The Q^{hol}-twist of the local Lie superalgebra underlying the $\mathcal{N} = (2,0)$ superconformal stress-tensor multiplet on $\mathbb{R}^6$ is quasi-isomorphic to the local Lie superalgebra $\mathcal{E}(3|6)_{\mathcal{R}}$ on $\mathbb{C}^3$. Here $\mathcal{R} = \mathcal{O}_{\mathbb{C}^3}^{\oplus 2}$, together with the standard identification $\wedge^2\mathcal{R} \cong \mathcal{K}_{\mathbb{C}^3}$, supplies the flat determinant data entering the definition of $\mathcal{E}(3|6)_{\mathcal{R}}$.*

Thus $E(3|6)$ is not merely an abstract enlargement of the residual finite-dimensional symmetry: it is the formal infinitesimal symmetry algebra of the flat local geometry obtained after twisting. The theorem motivates the expectation that the holomorphic twist of any $\mathcal{N} = (2,0)$ superconformal theory carries $\mathcal{E}(3|6)_{\mathcal{R}}$-currents. In this paper that action is constructed directly only for the free tensor multiplet.

3.2 Tanaka structures and the $E(3|6)$ symbol

We begin by fixing terminology for (super/graded) Tanaka structures on supermanifolds. Let

$$\mathfrak{m} = \mathfrak{m}_{-\ell} \oplus \cdots \oplus \mathfrak{m}_{-1} \tag{3.2}$$

be a negatively graded nilpotent Lie superalgebra generated by its degree -1 piece $\mathfrak{m}_{-1}$. We call the integer ℓ the *depth*. Let $G_0 \subset \mathrm{Aut}_{(0)}(\mathfrak{m})$ be a subgroup of its grading preserving automorphisms.

Definition 3.2. A *Tanaka structure of type* $(\mathfrak{m}, G_0)$ on a supermanifold M consists of:

(a) A filtration on the tangent bundle

$$\mathrm{T}_M = \mathrm{T}_M^{-\ell} \supset \cdots \supset \mathrm{T}_M^{-2} \supset \mathrm{T}_M^{-1}, \tag{3.3}$$

compatible with the Lie bracket of supervector fields on M. This bracket induces an $\mathcal{O}_M$-linear Lie bracket making

$$\operatorname{gr} \mathrm{T}_M = \bigoplus_{i=-\ell}^{-1} \mathrm{T}_M^i / \mathrm{T}_M^{i+1}. \tag{3.4}$$

into a weight-graded super Lie algebra in vector bundles on M. We require that this bundle be a locally trivial bundle of graded Lie superalgebras with fiber $\mathfrak{m}$.

(b) An *adapted frame* over an open set U is a choice of isomorphism $U \times \mathfrak{m} \to \operatorname{gr} \mathrm{T}_M|_U$. These fit together to define a principal $\mathrm{Aut}_{(0)}(\mathfrak{m})$-bundle of adapted frames, which we denote $\mathcal{A}\mathrm{ut}_{(0)}(\mathfrak{m})$. The further datum of a Tanaka structure is a reduction of structure group of this principal $\mathrm{Aut}_{(0)}(\mathfrak{m})$-bundle to G_0, which we denote by $\mathcal{F}\mathrm{r}(\mathfrak{m}, G_0)$.

We call the $\mathcal{O}_M$-linear bracket on $\operatorname{gr} \mathrm{T}_M$ the *Levi bracket*. In particular, the first Levi bracket associated to such a filtration is defined to be the canonical map

$$\mathfrak{L} \colon \mathrm{S}^2(\operatorname{gr}^{-1} \mathrm{T}_M) \to \operatorname{gr}^{-2} \mathrm{T}_M \tag{3.5}$$

One way to specify a filtration on the (super) tangent bundle is to provide a (super) distribution $\mathcal{D} \subset \mathrm{T}_M$. The filtration is then defined by setting $\mathrm{T}_M^{-1} = \mathcal{D}$ and recursively as

$$\mathrm{T}_M^{-j-1} = \mathrm{T}_M^{-j} + [\mathcal{D}, \mathrm{T}_M^{-j}]. \tag{3.6}$$

When this filtration is complete, we say that $\mathcal{D}$ is *bracket-generating;* a *bracket-generating Tanaka structure* is one whose filtration arises in this manner from the distribution T_M^{-1}. In the sequel, we will only be concerned with bracket-generating Tanaka structures induced by purely odd distributions.

Denote by $X = M^{\mathrm{red}}$ the underlying *reduced manifold;* it is defined at the level of structure sheaves by taking the quotient of $\mathcal{O}_M$ by the ideal $\mathcal{J}_M$ generated by all odd elements. As such, there is a canonical inclusion $i : X \to M$. When $\mathcal{D}$ is a purely odd distribution of maximal odd rank, restriction of even derivations to the reduced structure sheaf induces a canonical isomorphism

$$\rho\colon \ (\mathrm{T}_M/\mathscr{D})|_X \xrightarrow{\cong} \mathrm{T}_X. \tag{3.7}$$

A bracket-generating Tanaka structure of maximal odd rank with $G_0 = \mathrm{Aut}_{(0)}(\mathfrak{m})$ is called a *superconformal structure.*

We turn to our main example. Let A be a three-dimensional complex vector space and B a two-dimensional complex vector space. Fix an identification

$$\Omega_0\colon \ \wedge^2 B \xrightarrow{\cong} \wedge^3 A^*$$

and define

$$\mathfrak{m}_{-2} = A, \quad \mathfrak{m}_{-1} = \Pi(A^* \otimes B^*). \tag{3.8}$$

Let $\Omega_0^\vee\colon \ \wedge^3 A \longrightarrow \wedge^2 B^*$ be the dual isomorphism. Endow $\mathfrak{m} = \mathfrak{m}_{-2} \oplus \mathfrak{m}_{-1}$ with the symmetric super bracket

$$[\xi \otimes \alpha, \eta \otimes \beta] \overset{\text{def}}{=} \iota_{\xi\wedge\eta}\left((\Omega_0^\vee)^{-1}(\alpha \wedge \beta)\right) \in A = \mathfrak{m}_{-2}. \tag{3.9}$$

As our group G_0 we take all grading preserving automorphisms

$$G_0 = \mathrm{Aut}_{(0)}(\mathfrak{m}) = \{(g, h) \in \mathrm{GL}(A) \times GL(B) \mid \det h = \det g^{-1}\}. \tag{3.10}$$

Its Lie algebra is

$$\mathfrak{g}_0 = \mathrm{Der}_{(0)}(\mathfrak{m}) \cong \mathfrak{sl}(A) \oplus \mathfrak{sl}(B) \oplus \mathbb{C}. \tag{3.11}$$

Lemma 3.3. *There are natural identifications*

$$\mathrm{Aut}_{(0)}(\mathfrak{m}) \cong \left\{(g, h) \in GL(A) \times GL(B) \,\middle|\, \det(h) = \det(g)^{-1}\right\}, \tag{3.12}$$
$$\mathrm{Der}_{(0)}(\mathfrak{m}) \cong \{(a, b) \in \mathfrak{gl}(A) \oplus \mathfrak{gl}(B) \mid \mathrm{tr}(a) + \mathrm{tr}(b) = 0\}. \tag{3.13}$$

In particular,

$$\mathrm{Der}_{(0)}(\mathfrak{m}) \cong \mathfrak{sl}(A) \oplus \mathfrak{sl}(B) \oplus \mathbb{C}. \tag{3.14}$$

Proof. The displayed subgroup acts on $\mathfrak{m}_{-2} = A$ through g, and on $\mathfrak{m}_{-1} = \Pi(A^* \otimes B^*)$ through $(g^{-1})^* \otimes (h^{-1})^*$. It preserves the bracket (3.9) exactly when it preserves the determinant identification

$$\Omega_0\colon \ \wedge^2 B \longrightarrow \wedge^3 A^*,$$

which is the relation $\det(h) = \det(g)^{-1}$.

It remains to see that these are all the grading-preserving automorphisms. Let

$$q\colon A^* \otimes B^* \longrightarrow A, \qquad q(u) = \tfrac{1}{2}[u, u].$$

The projectivized zero locus of q is the locus of rank-one tensors, hence the Segre variety

$$\mathbb{P}(A^*) \times \mathbb{P}(B^*) \subset \mathbb{P}(A^* \otimes B^*).$$

Every graded automorphism of $\mathfrak{m}$ preserves this variety. Since its two factors have different dimensions, its projective automorphism group is $PGL(A^*) \times PGL(B^*)$. Lifting the two projective transformations and absorbing their scalar ambiguity shows that the action on $\mathfrak{m}_{-1}$ is of the form $(g^{-1})^* \otimes (h^{-1})^*$. Compatibility with q then fixes the action on $\mathfrak{m}_{-2}$ and imposes the determinant relation above. This proves the first identification. Differentiating that relation gives $\mathrm{tr}(a) + \mathrm{tr}(b) = 0$, and separating traceless and scalar parts gives the final decomposition. □

Definition 3.4. Let M be a complex supermanifold of dimension $3|6$. An $E(3|6)$*-structure* on M is a holomorphic geometric Tanaka structure of type $(\mathfrak{m}, G_0)$ (as in equations (3.8), (3.9), and (3.10)).

3.2.1. The next proposition gives a more explicit description of the geometric data needed for an $E(3|6)$-structure. To state it, we recall a construction used above. If $\mathcal{R}$ is a rank two vector bundle, and Ω is an isomorphism $\Omega\colon \wedge^2 \mathcal{R} \to \mathcal{K}_X$ then there is a natural composition

$$S^2(\Omega^1 \otimes \mathcal{R}^*) \xrightarrow{\wedge\otimes\wedge} \Omega^2_X \otimes \wedge^2\mathcal{R}^* \xrightarrow{\Omega} T_X \tag{3.15}$$

which we denote by $\wedge\Omega\wedge$.

Proposition 3.5. *Let $(M, \mathcal{D}, \mathcal{F}r(\mathfrak{m}, G_0))$ be an $E(3|6)$-structure on a $3|6$ complex supermanifold M and denote $X = M^{red}$. This data canonically determines:*

1. *a holomorphic rank two vector bundle $\mathcal{R}$ on X together with an isomorphism*

$$\Omega\colon \wedge^2 \mathcal{R} \xrightarrow{\cong} \mathcal{K}_X \quad ; \tag{3.16}$$

2. *an isomorphism*

$$\beta\colon \mathcal{D}|_X \xrightarrow{\cong} \Pi(\Omega^1_X \otimes \mathcal{R}^*) \tag{3.17}$$

such that the following diagram commutes

$$\begin{tikzcd} S^2(\mathcal{D}|_X) \arrow[r, "\mathfrak{L}"] \arrow[d, "S^2(\Pi^{-1}\beta)"] & (T_M/\mathcal{D})|_X \arrow[d, "\rho"] \\ S^2(\Omega^1_X \otimes \mathcal{R}^*) \arrow[r, "\wedge\Omega\wedge"] & T_X \end{tikzcd} . \tag{3.18}$$

Conversely, given such data $(\mathcal{D}, \mathcal{R}, \Omega, \beta)$ satisfying these properties there exists an $E(3|6)$ structure in the sense of definition 6.9.

Proof. Define the vector bundle $\mathcal{R}$ by the associated bundle construction

$$\mathcal{R} \stackrel{\text{def}}{=} \mathcal{F}r(\mathfrak{m}, G_0) \times^{G_0} B \tag{3.19}$$

where B is the defining representation as in equation (3.10). Similarly, by construction, the map ρ from equation (3.7) defines an isomorphism

$$\mathcal{F}r(\mathfrak{m}, G_0) \times^{G_0} A \cong \mathrm{gr}_{-2} T_M|_X \cong T_X. \tag{3.20}$$

Passing to the associated graded gives the isomorphism

$$\begin{aligned}\Omega\colon\ \wedge^2\mathcal{R} &= \mathcal{F}\mathrm{r}(\mathfrak{m}, G_0)\times^{G_0}\wedge^2 B\\ &= \mathcal{F}\mathrm{r}(\mathfrak{m}, G_0)\times^{G_0}\wedge^3 A^*\\ &\cong \mathcal{K}_X\end{aligned}$$

It is similar to produce β. Indeed, since $\mathfrak{m}_{-1} = \Pi(A^*\otimes B^*)$ we have

$$\beta\colon\ \mathrm{gr}_{-1}(\mathrm{T}_M)\cong \Pi\left(\Omega^1_X\otimes\mathcal{R}^*\right).$$

It is immediate to see that, locally, the map $\wedge\Omega\wedge$ is identified with the local $E(3|6)$ symbol bracket.

The converse direction is immediate. □

3.3 The flat $E(3|6)$-structure

Let $N_\mathfrak{m}$ be the connected, simply connected complex Lie supergroup integrating $\mathfrak{m}$. Since $\mathfrak{m}$ is two-step nilpotent, the exponential map identifies its underlying supermanifold with $\mathfrak{m}$, with multiplication

$$x\star y = x + y + \frac{1}{2}[x,y]. \tag{3.21}$$

Let $\mathcal{D}_{\text{flat}}\subset \mathrm{T}_{N_\mathfrak{m}}$ be the left-invariant distribution determined by $\mathfrak{m}_{-1}$:

$$(\mathcal{D}_{\text{flat}})_g = (L_g)_*\mathfrak{m}_{-1}. \tag{3.22}$$

Left translation identifies the graded symbol bundle with $N_\mathfrak{m}\times\mathfrak{m}$, and hence supplies the canonical reduction

$$N_\mathfrak{m}\times G_0\subset \mathrm{Iso}_{\mathrm{grLie}}\big(\mathfrak{m}, \mathrm{gr}(\mathrm{T}_{N_\mathfrak{m}})\big).$$

The resulting Tanaka structure is called the *flat $E(3|6)$-structure*. Its symbol algebra at every point is $\mathfrak{m}$.

The grading of $\mathfrak{m}$ determines a $\mathbb{C}^\times$-action

$$\delta_t|_{\mathfrak{m}_{-1}} = t, \qquad \delta_t|_{\mathfrak{m}_{-2}} = t^2. \tag{3.23}$$

Because the bracket is homogeneous, every δ_t is a Lie supergroup automorphism preserving $\mathcal{D}_{\text{flat}}$ and its canonical G_0-reduction.

Let $\widehat{N}_{\mathfrak{m},e}$ denote the formal completion at the identity.

Definition 3.6. The Lie superalgebra of formal infinitesimal automorphisms of the flat structure is

$$\widehat{\mathfrak{aut}}(N_\mathfrak{m}, \mathcal{D}_{\text{flat}}) = \left\{V\in \mathrm{Der}_{\mathrm{cts}}(\mathcal{O}_{\widehat{N}_{\mathfrak{m},e}})\ \middle|\ [V,\widehat{\Gamma}(\mathcal{D}_{\text{flat}})]\subseteq\widehat{\Gamma}(\mathcal{D}_{\text{flat}})\right\}. \tag{3.24}$$

Here $\widehat{\Gamma}(\mathcal{D}_{\text{flat}})$ is the module of formal sections of the completed distribution. Since $G_0 = \mathrm{Aut}_{(0)}(\mathfrak{m})$, preserving the distribution also preserves the induced G_0-structure.

We recall the algebraic construction that controls this symmetry algebra.

Definition 3.7. Let $\mathfrak{m} = \bigoplus_{j=-\ell}^{-1} \mathfrak{m}_j$ be a fundamental negatively graded Lie superalgebra and let $\mathfrak{g}_0 \subseteq \mathrm{Der}_{(0)}(\mathfrak{m})$. Set

$$\mathrm{prol}_j(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m}_j \quad (j < 0), \qquad \mathrm{prol}_0(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{g}_0.$$

For $k > 0$, define $\mathrm{prol}_k(\mathfrak{m}, \mathfrak{g}_0)$ recursively to be the space of homogeneous linear maps

$$\varphi\colon \mathfrak{m} \longrightarrow \bigoplus_{r<k} \mathrm{prol}_r(\mathfrak{m}, \mathfrak{g}_0)$$

such that $\varphi(\mathfrak{m}_j) \subseteq \mathrm{prol}_{j+k}(\mathfrak{m}, \mathfrak{g}_0)$ and, for homogeneous $u, v \in \mathfrak{m}$,

$$\varphi([u, v]) = [\varphi(u), v] + (-1)^{|\varphi||u|}[u, \varphi(v)]. \tag{3.25}$$

The *algebraic Tanaka prolongation* and its *completion* are respectively

$$\mathrm{prol}(\mathfrak{m}, \mathfrak{g}_0) = \bigoplus_{k\geq -\ell} \mathrm{prol}_k(\mathfrak{m}, \mathfrak{g}_0), \qquad \widehat{\mathrm{prol}}(\mathfrak{m}, \mathfrak{g}_0) = \prod_{k\geq -\ell} \mathrm{prol}_k(\mathfrak{m}, \mathfrak{g}_0). \tag{3.26}$$

Equivalently, $\mathrm{prol}(\mathfrak{m}, \mathfrak{g}_0)$ is the maximal transitive graded Lie superalgebra with negative part $\mathfrak{m}$ and degree-zero part $\mathfrak{g}_0$. Transitivity means that if X has nonnegative degree and $[X, \mathfrak{m}_{-1}] = 0$, then $X = 0$.

Proposition 3.8 (Flat realization)**.** *The dilation action gives a complete grading*

$$\widehat{\mathfrak{aut}}(N_{\mathfrak{m}}, \mathscr{D}_{\mathrm{flat}}) = \prod_{k\geq -2} \mathfrak{g}_k. \tag{3.27}$$

Under this grading, the map

$$\mathfrak{m} \longrightarrow \widehat{\mathfrak{aut}}(N_{\mathfrak{m}}, \mathscr{D}_{\mathrm{flat}}), \qquad x \longmapsto -x^{\mathrm{R}}, \tag{3.28}$$

identifies $\mathfrak{m}_j$ with $\mathfrak{g}_j$ for $j = -2, -1$, and commutator with the negative part gives a canonical identification

$$\mathfrak{g}_0 \cong \mathrm{Der}_{(0)}(\mathfrak{m}).$$

Moreover, there is an isomorphism of complete graded Lie superalgebras

$$\widehat{\mathfrak{aut}}(N_{\mathfrak{m}}, \mathscr{D}_{\mathrm{flat}}) \cong \widehat{\mathrm{prol}}\big(\mathfrak{m}, \mathrm{Der}_{(0)}(\mathfrak{m})\big). \tag{3.29}$$

Proof. Choose exponential coordinates z_1, z_2, z_3 on $\mathfrak{m}_{-2}$ and odd coordinates θ^A, $A = 1, \ldots, 6$, on $\mathfrak{m}_{-1}$. They have weights two and one, respectively. Formal coefficients are power series in the z_i and polynomials in the θ^A, so every formal vector field has a unique completed homogeneous expansion, with no weights below -2. The distribution-preserving equation is homogeneous, which proves (3.27).

Right translations preserve every left-invariant distribution. Since $[x^{\mathrm{R}}, y^{\mathrm{R}}] = -[x, y]^{\mathrm{R}}$, the map $x \mapsto -x^{\mathrm{R}}$ is a homomorphism. To see directly that it exhausts the negative terms, choose a basis q_A of $\mathfrak{m}_{-1}$ with $[q_A, q_B] = c_{AB}{}^i e_i$. The left- and right-invariant odd vector fields are

$$Q_A = \frac{\partial}{\partial \theta^A} + \frac{1}{2} c_{AB}{}^i \theta^B \frac{\partial}{\partial z_i}, \qquad \overline{Q}_A = \frac{\partial}{\partial \theta^A} - \frac{1}{2} c_{AB}{}^i \theta^B \frac{\partial}{\partial z_i}. \tag{3.30}$$

A homogeneous symmetry of weight -2 is a constant linear combination of the $\partial/\partial z_i$, while substitution in $[V, Q_A] \in \Gamma(\mathscr{D}_{\mathrm{flat}})$ shows that a weight -1 symmetry is a constant linear combination of the $\overline{Q}_A$. Thus the negative part is $\mathfrak{m}$.

A weight-zero symmetry acts by commutator on the negative part. The Jacobi identity shows that this action is a grading-preserving derivation, giving

$$\mathfrak{g}_0 \longrightarrow \mathrm{Der}_{(0)}(\mathfrak{m}).$$

If the action is trivial, the symmetry commutes with every right-invariant vector field. It is therefore left-invariant and determined by its value at the identity. That value must vanish because $T_e N_{\mathfrak{m}}$ has only weights -2 and -1, so the map is injective. Conversely, every graded derivation of $\mathfrak{m}$ exponentiates formally to graded automorphisms of $N_{\mathfrak{m}}$ preserving $\mathscr{D}_{\mathrm{flat}}$, which proves surjectivity.

It remains to identify the positive terms. We proceed inductively on $k > 0$, assuming that the components of degree less than k have already been identified with the corresponding prolongation components. For $V \in \mathfrak{g}_k$, define

$$\iota_k(V)(u) = [V, -u^{\mathrm{R}}], \qquad u \in \mathfrak{m}.$$

The Jacobi identity is precisely the derivation identity in (3.25), so

$$\iota_k : \mathfrak{g}_k \longrightarrow \mathrm{prol}_k\big(\mathfrak{m}, \mathrm{Der}_{(0)}(\mathfrak{m})\big).$$

This map is injective: a vector field in its kernel commutes with all right-invariant fields, hence is left-invariant; a nonzero left-invariant field has negative weight.

For surjectivity, filter the equation $[V, \Gamma(\mathscr{D}_{\mathrm{flat}})] \subseteq \Gamma(\mathscr{D}_{\mathrm{flat}})$ by weighted order. Once all terms of weights less than k have been fixed, the equation for the weight-k term is exactly (3.25). Thus every element of the k-th prolongation determines, recursively and uniquely, a homogeneous weight-k polynomial vector field preserving $\mathscr{D}_{\mathrm{flat}}$. This is the standard flat-realization argument for Tanaka prolongations and proves that ι_k is surjective. Taking the product over all weights gives (3.29). □

We can now identify the flat symmetry algebra.

Theorem 3.9. *The Lie superalgebra of formal infinitesimal automorphisms of the flat $E(3|6)$-structure is the exceptional linearly compact Lie superalgebra $E(3|6)$:*

$$\widehat{\mathfrak{aut}}(N_{\mathfrak{m}}, \mathscr{D}_{\mathrm{flat}}) \cong E(3|6). \tag{3.31}$$

Proof. By Proposition 3.8 and Lemma 3.3,

$$\widehat{\mathfrak{aut}}(N_{\mathfrak{m}}, \mathscr{D}_{\mathrm{flat}}) \cong \widehat{\mathrm{prol}}\big(\mathfrak{m}, \mathfrak{sl}(A) \oplus \mathfrak{sl}(B) \oplus \mathbb{C}\big).$$

Cheng and Kac identify the corresponding graded prolongation with $E(3|6)$; see [CK99, case (C4), §4.4]. Completing in the positive grading gives its linearly compact formal form. □

Remark 3.10. The preceding result concerns formal vector fields. Polynomial vector fields give the algebraic graded form of $E(3|6)$, while convergent holomorphic vector fields give its analytic form. Passing from the formal result to a statement about a sheaf of holomorphic infinitesimal automorphisms requires either the explicit analytic coordinate realization of $E(3|6)$ or a separate convergence argument.

3.4 A homogeneous $E(3|6)$-structure over $\mathbb{P}^3$

We next construct a compact homogeneous example. Let V be a four-dimensional complex vector space with a volume form, and let W be a two-dimensional complex symplectic vector space. The volume form equips $\wedge^2 V^\vee$ with its standard quadratic form and gives

$$\mathfrak{so}(\wedge^2 V^\vee) \cong \mathfrak{sl}(V).$$

Set

$$G = \mathrm{OSp}(\wedge^2 V^\vee \mid W^\vee), \qquad \mathfrak{g} = \mathfrak{osp}(\wedge^2 V^\vee \mid W^\vee).$$

Thus

$$\mathfrak{g}_{\bar 0} = \mathfrak{sl}(V) \oplus \mathfrak{sl}(W), \qquad \mathfrak{g}_{\bar 1} = \wedge^2 V^\vee \otimes W^\vee. \tag{3.32}$$

Fix a line $\ell \subset V$, put $Q = V/\ell$, and let

$$L_\ell = \wedge^2 Q^\vee \subset \wedge^2 V^\vee.$$

This is a maximal isotropic subspace. Define

$$\mathfrak{p}_{\bar 0} = \mathrm{stab}_{\mathfrak{sl}(V)}(\ell) \oplus \mathfrak{sl}(W), \tag{3.33}$$

$$\mathfrak{p}_{\bar 1} = L_\ell \otimes W^\vee, \tag{3.34}$$

and set $\mathfrak{p} = \mathfrak{p}_{\bar 0} \oplus \mathfrak{p}_{\bar 1}$.

Lemma 3.11. *The subspace $\mathfrak{p} \subset \mathfrak{g}$ is a Lie subsuperalgebra. Moreover, $\mathfrak{g}_{\bar 1}/\mathfrak{p}_{\bar 1} \subset \mathfrak{g}/\mathfrak{p}$ is invariant under the isotropy action of $\mathfrak{p}$.*

Proof. The even part preserves L_ℓ. If $u, v \in L_\ell$, then $q(u, v) = 0$, so the $\mathfrak{sl}(W)$-component of $[u \otimes \lambda, v \otimes \mu]$ vanishes. Its $\mathfrak{sl}(V)$-component is the orthogonal endomorphism $u \wedge v$, which preserves L_ℓ, and hence lies in $\mathrm{stab}_{\mathfrak{sl}(V)}(\ell)$. This proves that $\mathfrak{p}$ is closed under the bracket.

More generally, if $u \in L_\ell$, $v \in \wedge^2 V^\vee$, and $x \in L_\ell$, then

$$(u \wedge v)(x) = q(v, x)u - q(u, x)v = q(v, x)u \in L_\ell.$$

Thus $[\mathfrak{p}_{\bar 1}, \mathfrak{g}_{\bar 1}] \subseteq \mathfrak{p}_{\bar 0}$, which proves the isotropy-invariance of the indicated odd subspace of $\mathfrak{g}/\mathfrak{p}$. ◻

Let $P \subset G$ be the subsupergroup integrating $\mathfrak{p}$.

Proposition 3.12. *The homogeneous supermanifold*

$$M = G/P$$

carries a canonical G-invariant $E(3|6)$-structure. Its reduced space is $X = \mathbb{P}(V)$, and the rank-two bundle associated to its G_0-reduction is

$$\mathcal{R} = \mathcal{O}_{\mathbb{P}(V)}(-2) \otimes W, \qquad \det(\mathcal{R}) \cong K_{\mathbb{P}(V)}. \tag{3.35}$$

Proof. The reduced homogeneous space is

$$M^{\mathrm{red}} = G_{\bar{0}}/P_{\bar{0}} \cong SL(V)/\mathrm{Stab}(\ell) \cong \mathbb{P}(V).$$

At the base point $o = eP$, the even tangent space is

$$(\mathfrak{g}/\mathfrak{p})_{\bar{0}} \cong \mathfrak{sl}(V)/\mathrm{stab}_{\mathfrak{sl}(V)}(\ell) \cong \mathrm{Hom}(\ell, Q),$$

which is $T_{\mathbb{P}(V),[\ell]}$. The odd tangent space follows from the exact sequence

$$0 \longrightarrow \wedge^2 Q^\vee \longrightarrow \wedge^2 V^\vee \longrightarrow \ell^\vee \otimes Q^\vee \longrightarrow 0 :$$

namely,

$$(\mathfrak{g}/\mathfrak{p})_{\bar{1}} \cong \ell^\vee \otimes Q^\vee \otimes W^\vee.$$

Lemma 3.11 therefore gives a well-defined G-invariant rank $0|6$ distribution

$$\mathcal{D} = G \times_P (\mathfrak{g}_{\bar{1}}/\mathfrak{p}_{\bar{1}}) \subset G \times_P (\mathfrak{g}/\mathfrak{p}) = \mathrm{T}_M. \tag{3.36}$$

Globalizing the preceding description over $\mathbb{P}(V)$, the tautological sequence gives

$$\ell^\vee = \mathcal{O}_{\mathbb{P}(V)}(1), \qquad Q^\vee \cong \Omega^1_{\mathbb{P}(V)}(1).$$

Consequently

$$\mathcal{D}|_{\mathbb{P}(V)} \cong \Pi\big(\Omega^1_{\mathbb{P}(V)}(2) \otimes W^\vee\big) \cong \Pi\big(\Omega^1_{\mathbb{P}(V)} \otimes \mathcal{R}^\vee\big), \tag{3.37}$$

where $\mathcal{R}$ is given by (3.35). The chosen symplectic form trivializes $\det W$, while the volume form on V identifies $K_{\mathbb{P}(V)}$ with $\mathcal{O}_{\mathbb{P}(V)}(-4)$. Hence

$$\det(\mathcal{R}) = \mathcal{O}_{\mathbb{P}(V)}(-4) \otimes \det W \cong K_{\mathbb{P}(V)}.$$

It remains only to identify the Levi symbol. At o, it is induced by the odd-odd bracket

$$\mathrm{S}^2(\ell^\vee \otimes Q^\vee \otimes W^\vee) \longrightarrow \mathrm{Hom}(\ell, Q).$$

The bracket in $\mathfrak{osp}(6|2)$, together with the chosen volume and symplectic forms, takes the wedge product in the $Q^\vee$-factors and the wedge product in the $W^\vee$-factors. Under (3.37), this is exactly the map

$$\mathrm{S}^2(\Omega^1_{\mathbb{P}(V)} \otimes \mathcal{R}^\vee) \xrightarrow{\wedge\,\Omega\,\wedge} \mathrm{T}_{\mathbb{P}(V)}.$$

Since the construction is homogeneous, checking this at o suffices. Thus $\mathcal{D}$ has the standard $E(3|6)$ symbol and defines the claimed structure. $\square$

4 Free field realization of the exceptional symmetry

4.1 Free non-Lagrangian theories in the BV formalism

The field space of a Lagrangian field theory in the Batalin–Vilkovisky formalism is naturally equipped with a (local) (-1)-shifted symplectic structure. On the other hand, the six-dimensional superconformal theory is famously not Lagrangian, even in the abelian case. Its structure as a classical field theory is nevertheless encoded by (a slight generalization of) the BV formalism [SW23a]. We recall that structure here.

The shifted symplectic structure on the space of fields of a Lagrangian theory is already present at the level of just the free theory. A free theory, in the usual BV formalism, is encoded by a complex of vector bundles (E, Q) on spacetime together with a pairing

$$\omega \in \Gamma(M, E^* \otimes E^* \otimes \mathrm{Dens}_M)[-1]. \tag{4.1}$$

As indicated, this pairing has cohomological degree -1; it is required to satisfy $Q\omega = 0$, and is taken to be nondegenerate, and thus invertible, when restricted to any fiber over $x \in M$. The inverse defines a section

$$\begin{aligned} \pi = \omega^{-1} &\in \Gamma\left(M, E \otimes E \otimes \mathrm{Dens}^{-1}\right) \\ &= \Gamma\left(M, \mathrm{Hom}(E^!, E)\right). \end{aligned}$$

More generally, one can consider the sheaf $\mathrm{Diff}(E^!, E)$ of differential operators from sections of $E^!$ to sections of E. Notice that the differential Q induces a differential on this sheaf, which we will denote by the same letter.

The following definition encapsulates free, but non-Lagrangian, theories within the BV formalism.

Definition 4.1 (Compare [BY16, Definitions 2.30 and 2.34])**.** A (Poisson degenerate) *free BV theory* on a manifold M is an elliptic complex of vector bundles (E, Q) together with a graded skew-symmetric section

$$\pi \in \Gamma\left(M, \mathrm{Diff}(E^!, E)\right)[-1]$$

that satisfies $Q\pi = 0$.[2]

The section π allows one to define an important object in the BV formalism: the BV bracket on local functionals. We spell out this bracket at the level of the local functionals of the free theory. The construction mimics the usual definition of the Poisson bracket in classical mechanics. Given a local functional $F(\varphi)$ of the fields $\mathscr{E} = \Gamma(M, E)$ we can formally define the one-form valued local functional $\mathrm{d}F(\varphi)$. Contraction with π then defines a local vector field $X_F(\varphi)$ on the space of fields. The BV bracket between two local functionals is

$$\{F, G\} = X_F(G). \tag{4.2}$$

[2] Since we only consider cases in which π is a constant-coefficient Poisson bivector, the condition $[\pi, \pi] = 0$ is satisfied automatically.

For more details we refer to [BY16; CG21].

We will refer to π as the *(local)* 1*-shifted Poisson structure* of the theory. When formulated in terms of curvatures or field strengths, the $\mathcal{N} = (2,0)$ abelian tensor multiplet (and all of its twists) provide examples of such a structure [SW23a], as we will recall at greater length presently.

4.2 Holomorphic twists of the $\mathcal{N} = (2,0)$ theory

Neither the $\mathcal{N} = (2,0)$ superconformal theory nor its holomorphic twist admits a local Lagrangian description. In the supersymmetric theory, this arises due to the presence of a two-form field $B \in \Omega^2(\mathbf{R}^6)$ whose curvature $C = \mathrm{d}B$ is required to be self-dual: in Euclidean signature, it satisfies $\star C = \sqrt{-1}\,C$. In [SW23a], we show that the self-duality constraint means that the theory formulated in terms of the two-form field B is slightly degenerate: it admits a (-1)-shifted *presymplectic* structure. To place the theory into the context of the Poisson-degenerate BV theories defined above, we must formulate it in terms of the curvature C.

At a heuristic level, the dynamics of this field can be thought of as arising from the non-local action functional

$$\int C\mathrm{d}\mathrm{d}^{-1}C. \tag{4.3}$$

Since C, being a curvature, is necessarily closed, it locally admits a primitive form, and we can make sense of the resulting Euler–Lagrange equations. In [SW23a] we develop a more robust method and rigorously construct this theory as a Poisson-degenerate free BV theory with fundamental field C.

In any holomorphic twist of the $\mathcal{N} = (2,0)$ theory, there is a similar field. It is a complex two-form b with components of type $(0,2) + (1,1)$

$$\begin{aligned} \partial \mathrm{b}^{0,2} + \bar\partial \mathrm{b}^{1,1} &= 0 \\ \bar\partial \mathrm{b}^{0,2} &= 0. \end{aligned}$$

These equations are not variational in the usual sense. If we treat the curvature $\mathrm{c} = \partial \mathrm{b}$ as the fundamental field, such equations can be heuristically reproduced from the non-local Lagrangian

$$\int \mathrm{c}\bar\partial\partial^{-1}\mathrm{c}. \tag{4.4}$$

Again, we can make sense of the resulting Euler–Lagrange equations: Since c, being a holomorphic curvature, is necessarily ∂-closed, we can locally find a primitive for c. In [SW23a], we construct the twisted theory rigorously, again as a local Poisson-degenerate free BV theory with fundamental field c. We recall that result now.

We will not provide details of twisting or the extraction of the holomorphic twist from the fully supersymmetric theory; we refer to [SW23a] for details.

Definition 4.2 (generalization of [SW23a])**.** Let X be a complex threefold, $\mathcal{R}$ a rank two holomorphic vector bundle, and $\Omega\colon \wedge^2 \mathcal{R} \to \mathcal{K}_X$ an isomorphism. Define the complex of holomorphic super vector bundles on X:

$$\begin{array}{ccc} & \underline{-1} & \underline{0} \\ \underline{0} & \Omega^2_X \xrightarrow{\partial} & \Omega^3_X \\ \underline{1} & \mathcal{R} & \end{array} \tag{4.5}$$

which we will denote by $(\mathcal{V}^\bullet_{\mathcal{R}}, Q)$. We define the Poisson-degenerate free BV theory with sheaf of fields

$$\mathcal{V}^\bullet_{\mathcal{R}} \stackrel{\text{def}}{=} \Omega^{0,\bullet}(X, \mathcal{V}^\bullet_{\mathcal{R}})$$

differential $\overline{\partial} + Q$, and local (even) (-1)-shifted Poisson structure given on the even part by the holomorphic differential operator

$$\pi^{\underline{0}} = \partial \colon \Omega^{\leq 1}_X \to \Omega^{\geq 2}_X, \tag{4.6}$$

and on the odd part by the zeroth-order operator

$$\pi^{\underline{1}} = \Omega \colon \mathcal{R}^! \cong \mathcal{R}. \tag{4.7}$$

We will write Dolbeault-valued fields of the bundle in the first line of (4.5) with the letter c and in the bottom line ϕ. Some comments on the BV structure are warranted. The even part of the complex of fields is simply

$$\mathcal{V}^{\underline{0}}_{\mathcal{R}} = \Omega^{\geq 2,\bullet}(X)[1] \tag{4.8}$$

The differential Q acting is a sum $\overline{\partial} + \partial$. In particular, one has

$$(\mathcal{V}^{\underline{0}}_{\mathcal{R}})^! = \Omega^{\leq 1,\bullet}(X)[2]. \tag{4.9}$$

The holomorphic de Rham differential defines a natural map of degree $+1$

$$\partial \colon (\mathcal{V}^{\underline{0}}_{\mathcal{R}})^! \to \mathcal{V}^{\underline{0}}_{\mathcal{R}}. \tag{4.10}$$

The odd part of the complex of fields is

$$\mathcal{V}^{\underline{1}} = \Omega^{0,\bullet}(X, \mathcal{R})[1]. \tag{4.11}$$

Note that $(\mathcal{V}^{\underline{1}}_{\mathcal{R}})^! = \Omega^{0,\bullet}(X, \mathcal{R}^*)[2]$, so that the isomorphism Ω defines a natural map of degree $+1$

$$\Omega \colon (\mathcal{V}^{\underline{1}}_{\mathcal{R}})^! \xrightarrow{\cong} \mathcal{V}^{\underline{1}}_{\mathcal{R}}[1]. \tag{4.12}$$

Together, these two maps constitute π.

At the heuristic level of the discussion in the previous subsection, this free BV theory can be thought of as modelled by the non-local Lagrangian

$$\frac{1}{2}\int_X \mathrm{c}\overline{\partial}\partial^{-1}\mathrm{c} + \frac{1}{2}\int_X \Omega(\phi \wedge \overline{\partial}\phi). \tag{4.13}$$

The first term corresponds to the twist of the "$\mathcal{N} = (1,0)$ tensor multiplet." It has the anticipated form (4.4), reflecting the fact that the field c must be ∂-closed. There is also an ordinary, local, term in the Lagrangian, describing just the fields ϕ which originate from the $\mathcal{N} = (1,0)$ hypermultiplet (which is nondegenerate).

4.2.1. Polarized version. This section parallels 2.1.4. Namely, on the product manifold $X = C \times S$, where C is a complex curve and S a complex surface, there is a consistent way to lift the $\mathbf{Z} \times \mathbf{Z}/2$ grading of the free theory introduced above to a $\mathbf{Z}$-graded BV theory. As in section 2.1.4 we take $\mathcal{R} = \pi_C^* \mathcal{K}_C \oplus \pi_S^* \mathcal{K}_S$.

Definition 4.3. Let X be a complex threefold, $\mathcal{R}$ a rank two holomorphic vector bundle, and $\Omega\colon \wedge^2 \mathcal{R} \to \mathcal{K}_X$ an isomorphism. Define the complex of holomorphic super vector bundles on X:

$$\begin{array}{ccccc} \underline{-2} & & \underline{-1} & & \underline{0} \\ & & \Omega^2_X & \xrightarrow{\partial} & \Omega^3_X \\ \mathcal{O}_C \boxtimes \Omega^2_S & & & & \Omega^1_C \boxtimes \mathcal{O}_S \end{array} \tag{4.14}$$

which we will denote by $(\widetilde{\mathcal{V}}^\bullet, Q)$. We define the Poisson-degenerate free BV theory with sheaf of fields

$$\widetilde{\mathscr{V}}^\bullet \overset{\text{def}}{=} \Omega^{0,\bullet}(X, \widetilde{\mathcal{V}}^\bullet)$$

differential $\bar{\partial} + Q$, and local (-1)-shifted Poisson structure given on the first line by the holomorphic differential operator

$$\pi^{\underline{0}} = \partial\colon \Omega_X^{\leq 1} \to \Omega_X^{\geq 2}, \tag{4.15}$$

On the second line the (-1)-shifted symplectic structure is prescribed by Serre duality.

As before, we write the closed two-form fields by the symbol c. Now, in the polarized version, the odd field odd ϕ has split into two fields

$$\gamma \in \Omega^{1,\bullet}(C) \mathbin{\widehat{\otimes}} \Omega^{0,\bullet}(S), \quad \beta \in \Omega^{0,\bullet}(C) \mathbin{\widehat{\otimes}} \Omega^{2,\bullet}(S)[2]. \tag{4.16}$$

The non-local form of the action is almost identical:

$$\frac{1}{2}\int_X \mathrm{c}\bar{\partial}\partial^{-1}\mathrm{c} + \int_X \beta \wedge \bar{\partial}\gamma. \tag{4.17}$$

4.3 Recollection: symmetries in the BV formalism

Deformation theory and dg Lie algebras [Hin97; Lur11] have a longstanding and interconnected relationship. To every classical (Lagrangian) field theory within the BV formalism, one can associate a universal dg Lie algebra which controls the derived deformation theory of the theory itself: all of its symmetries, deformations, obstructions, and so on.

Let $\mathscr{E}$ denote the sheaf of fields of a BV theory, and let $\mathcal{O}_{\mathrm{loc}}(\mathscr{E})$ denote its sheaf of local functionals. As we have already explained, the BV bracket $\{-,-\}$ endows $\mathcal{O}_{\mathrm{loc}}(\mathscr{E})$ with the structure of a sheaf of graded shifted Lie algebras.[3] After shifting by one to account for the degree of the BV bracket, $\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]$ is hence a graded Lie algebra. Together with the classical BRST differential, which we denote Q, the object

$$(\mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1], Q, \{-,-\}) \tag{4.18}$$

is a dg Lie algebra.

This dg Lie algebra governs the deformation theory of the BV theory $\mathcal{E}$. To see that this makes sense, note that a Maurer–Cartan element is a degree-zero local functional S' satisfying the equation

$$QS' + \frac{1}{2}\{S', S'\} = \frac{1}{2}\{S + S', S + S'\} = 0.$$

(Here S is the BV action functional, so that $Q = \{S, -\}$.) Thus a Maurer–Cartan element is precisely a deformation of the action functional that satisfies the classical master equation. Following the general yoga of derived deformation theory, one sees that $H^{-1}(\mathcal{O}_{\mathrm{loc}}(E))$ encodes first-order symmetries of the theory, $H^0(\mathcal{O}_{\mathrm{loc}}(E))$ encodes first-order deformations, and $H^1(\mathcal{O}_{\mathrm{loc}}(E))$ is the home of obstructions, or anomalies.

Suppose that $\mathcal{L}$ is a local dg Lie superalgebra. A strict $\mathcal{L}$-symmetry on a classical field theory is a morphism of dg Lie algebras

$$\mathbf{J} \colon \mathcal{L} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})[-1]. \tag{4.19}$$

For each section μ of $\mathcal{L}$ the corresponding local functional $\mathbf{J}_\mu$ is a local version of the Noether current associated to the symmetry μ. More generally, one should allow for L_∞-morphisms of this type; such morphisms are equivalently described as solutions to the *equivariant* classical master equation

$$\mathrm{d}_{\mathcal{L}}\mathbf{J} + Q\mathbf{J} + \frac{1}{2}\{\mathbf{J}, \mathbf{J}\}_{BV} = 0, \tag{4.20}$$

where $\mathrm{d}_{\mathcal{L}}$ is the Chevalley–Eilenberg differential associated to $\mathcal{L}$.

We will provide a solution to the equivariant master equation in the case that $\mathcal{L}$ is $\mathcal{E}(3|6)_{\mathcal{R}}$ introduced in 2 and where $\mathcal{E}$ is the Poisson-degenerate BV theory $\mathcal{V}_{\mathcal{R}}$ which we have just introduced. In our situation, the Chevalley–Eilenberg differential $\mathrm{d}_{\mathcal{L}}$ has two terms: one coming from the differential underlying the local dg Lie algebra and the second coming from the local bracket.

4.4 Exceptional symmetry

We proceed to construct local Noether currents in the holomorphic twist of the free $\mathcal{N} = (2,0)$ theory for the exceptional local Lie superalgebra $\mathcal{E}$ defined in section 2. We use the notations and conventions from that section. Throughout, $|x|$ denotes the Dolbeault degree.

[3] In physics language, the grading is by the ghost number.

4.4.1. We first consider the even symmetries $\mathscr{E}(3|6)^0_{\mathcal{R}}$. It is rather immediate how

$$D \in \mathscr{E}(3|6)^0_{\mathcal{R}} = \Omega^{0,\bullet}(X, \mathcal{A}t_\Omega(\mathcal{R}))$$

acts on the fields of the free theory $\mathscr{V}$. The differential operator D acts on the even fields through its associated vector field and on the odd fields by definition. The problem is if we can find local Noether currents, or Hamiltonians, for such symmetries according to the BV Poisson structure that we have just introduced.

Define

$$\mathbf{T}^{(1)}_D = \frac{1}{2}\int_X \mathrm{c} \wedge \iota_{\#D}\mathrm{c} + \frac{1}{2}\int_X \Omega(\phi \wedge D\phi) \in \mathcal{O}_{\mathrm{loc}}(\mathscr{V})[-1]. \tag{4.21}$$

It is immediate to see the linear term in the equivariant master equation for the even symmetries:

$$(\bar\partial + Q)\mathbf{T}^{(1)}_D + \mathbf{T}^{(1)}_{\bar\partial D} = 0. \tag{4.22}$$

Indeed, the $\bar\partial$-parts are tautological and the Q-part vanishes by type reasons.

The BV bracket against the c-part of $\mathbf{T}^{(1)}_D$ generates the infinitesimal transformation

$$\mathrm{c} \longmapsto \partial\iota_\mu \mathrm{c}.$$

On the other hand, the natural vector-field action on c is through Lie derivative $L_\mu \mathrm{c}$. By Cartan's formula,

$$\partial\iota_\mu + (-1)^{|\mu|}\iota_\mu\partial = L_\mu,$$

we see the difference between the Hamiltonian action and the bare vector-field action is

$$\partial\iota_\mu \mathrm{c} - L_\mu \mathrm{c} = -(-1)^{|\mu|}\iota_\mu\partial\mathrm{c}. \tag{4.23}$$

Notice that when c is identically ∂-closed that the two actions agree. But, we are merely resolving the condition that c be ∂-closed. We will exhibit explicit homotopy correction terms which combined to result in a solution to the master equation.

4.4.2. The even stress tensor. Recall that $\bar\partial + Q$ is the differential underlying the complex of vector bundles $\mathscr{V}^\bullet$ describing the free theory. Here Q is a holomorphic differential operator which vanishes on the ϕ fields and $Q\mathrm{c} = \partial\mathrm{c}$ when c is a form of type $(2, \bullet)$ and zero otherwise.

We define the following L_∞ local current

$$\mathbf{T}^{\underline{0}} \overset{\text{def}}{=} \frac{1}{2}\int_X \mathrm{c} \wedge e^{\iota_{\#D}}(1 - \iota_{\#D}P_3)\mathrm{c} + \frac{1}{2}\int_X \Omega(\phi \wedge D\phi) \tag{4.24}$$

where $P_3 \colon \Omega^{\geq 2,\bullet}_X \to \Omega^{3,\bullet}_X[-1]$ is projection onto the $(3, \bullet)$ component. Notice that the term linear in D is precisely $\mathbf{T}^{(1)}_D$ in (4.21).

Proposition 4.4. *The local current $\mathbf{T}^{\underline{0}}$ satisfies the $\mathscr{E}(3|6)^0_{\mathcal{R}}$ equivariant classical master equation*

$$\mathrm{d}^{\underline{0}}_{CE}\mathbf{T}^{\underline{0}} + Q\mathbf{T}^{\underline{0}} + \frac{1}{2}\{\mathbf{T}^{\underline{0}}, \mathbf{T}^{\underline{0}}\} = 0. \tag{4.25}$$

where $\mathrm{d}^{\underline{0}}_{CE}$ is the CE differential for the local dg Lie algebra $\mathscr{E}(3|6)^0_{\mathcal{R}}$.

Proof. We first consider the c-part. Split the full Dolbeault–de Rham complex as

$$\Omega_X^{\bullet,\bullet} = E \oplus K[-2], \qquad E = \Omega_X^{\leq 1,\bullet}, \qquad K = \Omega_X^{\geq 2,\bullet},$$

and denote the two projections by P_E and P_K. Denote $A = \iota_{\#D}$. On a threefold one has the operator identities

$$P_K e^A\big|_K = 1 + AP_3, \qquad (AP_3)^2 = 0.$$

It follows that

$$T \overset{\text{def}}{=} e^A(1 - AP_3)\big|_K = 1 + B, \qquad B \overset{\text{def}}{=} P_E T\colon K \longrightarrow E. \tag{4.26}$$

Thus $e^A K = \mathrm{Graph}(B)$. If $\langle k, e\rangle = \int_X k \wedge e$ denotes the shifted wedge pairing, then the c-part of the current $\mathbf{T}^{\underline{0}}$ is

$$I_{\mathrm{c}} = \frac{1}{2}\langle \mathrm{c}, B\mathrm{c}\rangle. \tag{4.27}$$

Indeed, the K-component in (4.26) does not contribute since the wedge of two elements of K has holomorphic degree at least four. Also, A is an even derivation in the total CE–de Rham complex, so e^A is an algebra automorphism. Since $K \wedge K = 0$ in holomorphic degree three, the graph $e^A K$ is Lagrangian. Equivalently, B is cyclic for the shifted pairing, as required for (4.27).

We now compute the two terms in the master equation directly from the BV bracket. Relative to $E \oplus K$, write the Dolbeault–de Rham differential in the form

$$\begin{pmatrix} D_E & 0 \\ \omega & D_K \end{pmatrix}, \qquad \omega\colon \Omega_X^{1,\bullet} \xrightarrow{\partial} \Omega_X^{2,\bullet}. \tag{4.28}$$

The crossing term ω is precisely the Poisson operator defining the BV bracket on the c fields. Cyclicity of B gives $\mathrm{d}I_{\mathrm{c}} = B\mathrm{c}$, and hence the Hamiltonian vector field is $X_{I_{\mathrm{c}}} = \omega B\mathrm{c}$. Therefore, using the definition $\{F, G\} = X_F(G)$ and the shifted integration-by-parts convention,

$$\frac{1}{2}\{I_{\mathrm{c}}, I_{\mathrm{c}}\} = -\frac{1}{2}\langle \mathrm{c}, B\omega B\mathrm{c}\rangle, \tag{4.29}$$

$$(\mathrm{d}^{0}_{CE} + Q)I_{\mathrm{c}} = \frac{1}{2}\left\langle \mathrm{c}, \left(\mathrm{d}^{0}_{CE}B + D_E B - BD_K\right)\mathrm{c}\right\rangle. \tag{4.30}$$

Consequently the c-part of the classical master equation is exactly the operator identity

$$\mathrm{d}^{0}_{CE}B + D_E B - BD_K - B\omega B = 0. \tag{4.31}$$

It remains to verify (4.31). Let $\mathcal{D}$ denote the sum of the CE, Dolbeault, and holomorphic de Rham differentials on CE cochains with values in $\Omega_X^{\bullet,\bullet}$. By the definition of the CE differential and the fact that the anchor preserves brackets,

$$[\mathrm{d}^{0}_{CE} + \bar{\partial}, A] = -\frac{1}{2}\iota_{[\#D,\#D]}.$$

Together with the Cartan identities, this gives the finite conjugation formula

$$e^{-A}\mathcal{D}e^{A} = \mathcal{D} + L_{\#D}. \tag{4.32}$$

In fact, in the Baker–Campbell–Hausdorff expansion the terms after the second commutator vanish, while

$$[\partial, A] = L_{\#D}, \qquad \frac{1}{2}[L_{\#D}, A] = \frac{1}{2}\iota_{[\#D,\#D]}.$$

The right-hand side of (4.32) preserves K. Hence $e^A K$, or equivalently $\mathrm{Graph}(B)$, is $\mathcal{D}$-invariant. Applying $\mathcal{D}$ to $(B\mathrm{c}, \mathrm{c})$ and using (4.28), graph invariance says

$$\mathrm{d}^0_{CE}B + D_E B = B(D_K + \omega B),$$

which is (4.31). Equations (4.29)–(4.31) prove the c-part of the classical master equation directly.

The exponential in (4.26) is finite. Inside the quadratic functional it gives

$$I_{\mathrm{c}} = \frac{1}{2}\left\langle \mathrm{c}, \left(A + A^2\left(\frac{1}{2} - P_3\right) - \frac{1}{3}A^3\right)\mathrm{c}\right\rangle,$$

because $A^4 = 0$ and $A^3P_3 = A^3$ on K. Thus these are precisely the linear, quadratic, and cubic homotopies, with no higher terms.

Finally, the ϕ-part is the ordinary quadratic Hamiltonian for the action of D on $\mathcal{R}$. Since this action preserves Ω, the definition of the BV bracket gives

$$\mathrm{d}^0_{CE}I_\phi + \frac{1}{2}\{I_\phi, I_\phi\} = 0, \qquad QI_\phi = 0.$$

The mixed bracket vanishes because the Poisson structure is block diagonal with respect to the c and ϕ fields. Combining the two sectors proves the result. □

4.4.3. Odd and mixed components. We now add the Taylor components containing odd inputs. For homogeneous $D, D' \in \mathscr{E}(3|6)^{\underline{0}}_{\mathcal{R}}$ and $\alpha \in \mathscr{E}(3|6)^{\underline{1}}_{\mathcal{R}}$, write $\mu = \#D$ and $\mu' = \#D'$, and define

$$\mathbf{T}^{(1)}_{\alpha} \stackrel{\text{def}}{=} \int_X \mathrm{c}(\alpha, \phi), \tag{4.33}$$

$$\mathbf{T}^{(2)}_{D;\alpha} \stackrel{\text{def}}{=} \int_X (\iota_\mu \mathrm{c})(\alpha, \phi), \tag{4.34}$$

$$\mathbf{T}^{(3)}_{D,D';\alpha} \stackrel{\text{def}}{=} -2\int_X (\iota_\mu \iota_{\mu'} \mathrm{c})(\alpha, \phi). \tag{4.35}$$

When all inputs are even, $\mathbf{T}^{(k)}_{D_1,\ldots,D_k}$ denotes the polarized arity-k component of $\mathbf{T}^{\underline{0}}$. All remaining components containing odd inputs are zero:

$$\mathbf{T}^{(2)}_{\alpha,\beta} = \mathbf{T}^{(3)}_{D;\alpha,\beta} = \mathbf{T}^{(3)}_{\alpha,\beta,\gamma} = 0. \tag{4.36}$$

The linear equation for (4.33) is immediate:

$$\bar{\partial}\mathbf{T}^{(1)}_{\alpha} + \mathbf{T}^{(1)}_{\bar{\partial}\alpha} = 0. \tag{4.37}$$

It remains to check the nonlinear equations, which we organize by the parities of their inputs.

Lemma 4.5 (One even and one odd input)**.** *Let $D \in \mathscr{E}(3|6)^0_{\mathcal{R}}$ and $\alpha \in \mathscr{E}(3|6)^1_{\mathcal{R}}$ be homogeneous. Then*

$$(\bar\partial + Q)\mathbf{T}^{(2)}_{D;\alpha} + \mathbf{T}^{(2)}_{\bar\partial D;\alpha} + (-1)^{|D|}\mathbf{T}^{(2)}_{D;\bar\partial\alpha} = \mathbf{T}^{(1)}_{[D,\alpha]} - \{\mathbf{T}^{(1)}_D, \mathbf{T}^{(1)}_\alpha\}. \tag{4.38}$$

Proof. The $\bar\partial$-terms are tautological, so it remains to check the Q-term. Since $Q\phi = 0$ and $Q\mathrm{c} = \partial\mathrm{c}$ we have

$$Q\mathbf{T}^{(2)}_{D;\alpha} = \int_X (\iota_\mu\partial\mathrm{c})(\alpha, \phi). \tag{4.39}$$

We match this with the right hand side. Indeed, using Cartan's formula again we get

$$\{\mathbf{T}^{(1)}_D, \mathbf{T}^{(1)}_\alpha\} = \mathbf{T}^{(1)}_{[D,\alpha]} - \int_X (\iota_\mu\partial\mathrm{c})(\alpha, \phi). \tag{4.40}$$

This is precisely (4.38). □

Lemma 4.6 (Two even and one odd input)**.** *Let $D, D' \in \mathscr{E}(3|6)^0_{\mathcal{R}}$ and $\alpha \in \mathscr{E}(3|6)^1_{\mathcal{R}}$ be homogeneous. Then*

$$\begin{aligned}(\bar\partial + Q)\mathbf{T}^{(3)}_{D,D';\alpha} &+ \mathbf{T}^{(3)}_{\bar\partial D,D';\alpha} + (-1)^{|D|}\mathbf{T}^{(3)}_{D,\bar\partial D';\alpha} + (-1)^{|D|+|D'|}\mathbf{T}^{(3)}_{D,D';\bar\partial\alpha} \\ &= \mathbf{T}^{(2)}_{[D,D'];\alpha} - \mathbf{T}^{(2)}_{D;[D',\alpha]} + (-1)^{|D||D'|}\mathbf{T}^{(2)}_{D';[D,\alpha]} \\ &\quad - \{\mathbf{T}^{(1)}_D, \mathbf{T}^{(2)}_{D';\alpha}\} + (-1)^{|D||D'|}\{\mathbf{T}^{(1)}_{D'}, \mathbf{T}^{(2)}_{D;\alpha}\} \\ &\quad - (-1)^{(|\alpha|+1)(|D|+|D'|)}\{\mathbf{T}^{(1)}_\alpha, \mathbf{T}^{(2)}_{D,D'}\}.\end{aligned} \tag{4.41}$$

Proof. The $\bar\partial$-terms are tautological, so we focus on the Q-term and the BV brackets. Write

$$K_{\mu;\alpha} \overset{\text{def}}{=} \int_X (\iota_\mu\mathrm{c})(\alpha, \phi), \qquad K_{\mu,\mu';\alpha} \overset{\text{def}}{=} \int_X (\iota_\mu\iota_{\mu'}\mathrm{c})(\alpha, \phi).$$

it is enough to show that the right-hand side of (4.41) is $-2QK_{\mu,\mu';\alpha}$.

Since $Q\mathrm{c} = \partial\mathrm{c}$ and $Q\phi = 0$, we have

$$QK_{\mu,\mu';\alpha} = \int_X (\iota_\mu\iota_{\mu'}\partial\mathrm{c})(\alpha, \phi). \tag{4.42}$$

Next, compute the two brackets involving even Hamiltonians and mixed quadratic terms. Cartan's formula gives

$$\{\mathbf{T}^{(1)}_D, K_{\mu';\alpha}\} = K_{[\mu,\mu'];\alpha} + K_{\mu';[D,\alpha]} - \int_X (\iota_{\mu'}\iota_\mu\partial\mathrm{c})(\alpha, \phi). \tag{4.43}$$

Similarly, with D and D' exchanged,

$$\{\mathbf{T}^{(1)}_{D'}, K_{\mu;\alpha}\} = K_{[\mu',\mu];\alpha} + K_{\mu;[D',\alpha]} - \int_X (\iota_\mu\iota_{\mu'}\partial\mathrm{c})(\alpha, \phi). \tag{4.44}$$

Next, compute the bracket of the odd Hamiltonian with the even-even homotopy:

$$\{\mathbf{T}^{(1)}_{\alpha}, \mathbf{T}^{(2)}_{D,D'}\} = -\int_X (\iota_{\mu}\iota_{\mu'}\partial \mathrm{c})(\alpha, \phi). \tag{4.45}$$

We now substitute (4.43), (4.44), and (4.45) into the right-hand side of (4.41). The terms

$$\mathbf{T}^{(2)}_{[D,D'];\alpha} - \mathbf{T}^{(2)}_{D;[D',\alpha]} + (-1)^{|D||D'|}\mathbf{T}^{(2)}_{D';[D,\alpha]}$$

cancel term-by-term with the corresponding terms coming from

$$-\{\mathbf{T}^{(1)}_{D}, \mathbf{T}^{(2)}_{D';\alpha}\} + (-1)^{|D||D'|}\{\mathbf{T}^{(1)}_{D'}, \mathbf{T}^{(2)}_{D;\alpha}\}.$$

The remaining terms are therefore only the two Cartan homotopy terms

$$\int_X (\iota_{\mu'}\iota_{\mu}\partial \mathrm{c})(\alpha, \phi) - (-1)^{|D||D'|}\int_X (\iota_{\mu}\iota_{\mu'}\partial \mathrm{c})(\alpha, \phi).$$

By graded anticommutativity of contractions, this is $2QK_{\mu',\mu;\alpha} = Q\mathbf{T}^{(3)}_{D,D';\alpha}$. □

Lemma 4.7 (Two odd inputs)**.** *Let $\alpha, \beta \in \mathscr{E}(3|6)^{1}_{\mathcal{R}}$ be homogeneous. Then*

$$\{\mathbf{T}^{(1)}_{\alpha}, \mathbf{T}^{(1)}_{\beta}\} = \mathbf{T}^{(1)}_{[\alpha,\beta]}. \tag{4.46}$$

Proof. An odd symmetry α determines a bundle map

$$m_{\alpha} \colon \mathcal{R} \to \Omega^1_X, \qquad r \mapsto \alpha(r).$$

We also write $m^{\dagger}_{\alpha} \colon \Omega^2_X \to \mathcal{R}$ for its adjoint with respect to the pairing defined by Ω:

$$\Omega(r, m^{\dagger}_{\alpha}\eta) = \eta \wedge m_{\alpha}(r). \tag{4.47}$$

Thus $\mathbf{T}^{(1)}_{\alpha} = \int_X \mathrm{c} \wedge m_{\alpha}(\phi)$. By the definition of the odd-odd bracket in $\mathscr{E}(3|6)_{\mathcal{R}}$, the even element $[\alpha, \beta] \in \mathcal{A}\mathrm{t}_{\Omega}(\mathcal{R})$ acts by

$$\iota_{\#[\alpha,\beta]}\mathrm{c} = (m_{\alpha}m^{\dagger}_{\beta} + m_{\beta}m^{\dagger}_{\alpha})\mathrm{c}, \tag{4.48}$$

$$[\alpha, \beta]\phi = (m^{\dagger}_{\alpha}\partial m_{\beta} + m^{\dagger}_{\beta}\partial m_{\alpha})\phi. \tag{4.49}$$

The BV bracket splits according to which fields are contracted:

$$\{\mathbf{T}^{(1)}_{\alpha}, \mathbf{T}^{(1)}_{\beta}\} = I_{\mathrm{cc}} + I_{\phi\phi}, \tag{4.50}$$

where the subscripts record the contracted fields. Using (4.47) and graded symmetrizing in α, β gives

$$I_{\mathrm{cc}} = \frac{1}{2}\int_X \Omega\big(\phi \wedge (m^{\dagger}_{\alpha}\partial m_{\beta} + m^{\dagger}_{\beta}\partial m_{\alpha})\phi\big), \tag{4.51}$$

$$I_{\phi\phi} = \frac{1}{2}\int_X \mathrm{c} \wedge (m_{\alpha}m^{\dagger}_{\beta} + m_{\beta}m^{\dagger}_{\alpha})\mathrm{c}. \tag{4.52}$$

By (4.48) and (4.49), these are respectively the ϕ- and c-quadratic terms in $\mathbf{T}^{(1)}_{[\alpha,\beta]}$. This proves (4.46). □

Lemma 4.8 (One even and two odd inputs)**.** *Let $D \in \mathscr{E}(3|6)^0_{\mathcal{R}}$ and $\alpha, \beta \in \mathscr{E}(3|6)^1_{\mathcal{R}}$ be homogeneous. Then*

$$\begin{aligned}\mathbf{T}^{(2)}_{D;[\alpha,\beta]} - (-1)^{|D||\alpha|}\{\mathbf{T}^{(1)}_{\alpha}, \mathbf{T}^{(2)}_{D;\beta}\}& \\ + (-1)^{|\beta|(|D|+|\alpha|)}\{\mathbf{T}^{(1)}_{\beta}, \mathbf{T}^{(2)}_{D;\alpha}\} &= 0.\end{aligned} \tag{4.53}$$

Proof. The two BV brackets in (4.53) are

$$\{\mathbf{T}^{(1)}_{\alpha}, \mathbf{T}^{(2)}_{D;\beta}\} = \int_X (\iota_\mu \partial m_\alpha(\phi))(\beta, \phi) + \int_X (\iota_\mu \mathrm{c})(\beta, m^\dagger_\alpha \mathrm{c}), \tag{4.54}$$

$$\{\mathbf{T}^{(1)}_{\beta}, \mathbf{T}^{(2)}_{D;\alpha}\} = \int_X (\iota_\mu \partial m_\beta(\phi))(\alpha, \phi) + \int_X (\iota_\mu \mathrm{c})(\alpha, m^\dagger_\beta \mathrm{c}). \tag{4.55}$$

Substitute these expressions in (4.53). The adjointness relation (4.47), with the displayed Koszul signs, combines the two brackets into

$$\int_X (\iota_\mu \mathrm{c})([\alpha, \beta], \phi) = \mathbf{T}^{(2)}_{D;[\alpha,\beta]}.$$

$\square$

Lemma 4.9 (Three odd inputs)**.** *Let $\alpha, \beta, \gamma \in \mathscr{E}(3|6)^1_{\mathcal{R}}$ be homogeneous. Then*

$$\mathrm{Alt}_{\alpha,\beta,\gamma}\, \mathbf{T}^{(2)}_{[\alpha,\beta];\gamma} = 0. \tag{4.56}$$

Proof. The identity is pointwise and purely algebraic. Fix $x \in X$ and write

$$A = T_x X, \qquad B = \mathcal{R}_x.$$

The odd symbol is $A^* \otimes B^*$, so it is enough to check decomposable elements

$$\alpha = a \otimes u, \qquad \beta = b \otimes v, \qquad \gamma = c \otimes w,$$

where $a, b, c \in A^*$ and $u, v, w \in B^*$.

By definition of the odd-odd bracket, the anchor of $[\alpha, \beta]$ is the vector field corresponding, under contraction with the volume form on A, to

$$(a \wedge b)\, \omega(u, v) \in \wedge^2 A^*,$$

where ω is the symplectic pairing on B^* induced by $\Omega : \wedge^2 \mathcal{R} \cong K_X$. Therefore, for any local c-field and ϕ-field, the value of the integrand of $\mathbf{T}^{(2)}_{[\alpha,\beta];\gamma}$ at this point is proportional to

$$a \wedge b \wedge c\ \omega(u, v)\, w(\phi).$$

Hence the graded alternating sum over α, β, γ is proportional to

$$a \wedge b \wedge c\,\big(\omega(u, v) w(\phi) + \omega(v, w) u(\phi) + \omega(w, u) v(\phi)\big).$$

The expression in parentheses vanishes because B^* is two-dimensional symplectic: for any $u, v, w \in B^*$,

$$\omega(u, v) w + \omega(v, w) u + \omega(w, u) v = 0.$$

Thus the integrand in (4.56) vanishes pointwise. $\square$

4.4.4. The full equivariant current.

Theorem 4.10. *For any threefold X and holomorphic vector bundle $\mathcal{R}$ on X with $\wedge^2\mathcal{R} \cong K_X$, the Taylor components above define an L_∞ local current*

$$\mathbf{T} = \mathbf{T}^{(1)} + \mathbf{T}^{(2)} + \mathbf{T}^{(3)} : \mathscr{E}(3|6)_{\mathcal{R}} \rightsquigarrow \mathcal{O}_{\mathrm{loc}}(\mathscr{V})[-1]. \tag{4.57}$$

It solves the equivariant classical master equation and hence defines a $\mathscr{E}(3|6)_{\mathcal{R}}$-symmetry of the free Poisson-degenerate BV theory $\mathscr{V}$.

Proof. Proposition 4.4 proves the equation when all inputs are even. Equation (4.37) proves its linear odd part. Lemmas 4.5 and 4.6 prove the components with one odd input, while Lemmas 4.7, 4.8, and 4.9 prove those with at least two odd inputs. These exhaust the nonzero Taylor components listed in (4.36), and therefore prove the full equivariant classical master equation. □

4.4.5. Polarized stress tensor.

We finally record the stress tensor for the polarized theory of Definition 4.3. Write

$$\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_{-1}, \qquad \mathcal{R}_1 = \mathcal{K}_C \boxtimes \mathcal{O}_S, \qquad \mathcal{R}_{-1} = \mathcal{O}_C \boxtimes \mathcal{K}_S,$$

and correspondingly write the regraded field as $\phi = \gamma+\beta$, with γ valued in $\mathcal{R}_1$ and β valued in $\mathcal{R}_{-1}[2]$, as in Paragraph 4.2.1. A section of $\widetilde{\mathcal{E}}(3|6)$ coming from the two summands of $\mathcal{E}(3|6)_{\mathcal{R}}$ will be written

$$D = D_{-2} + D_0 + D_2, \qquad \alpha = \alpha_{-1} + \alpha_1.$$

Here D_i has degree i, α_j has degree j, and $\mu = \#D_0$. The components $D_{\pm 2}$ have zero anchor and exchange the two lines in $\mathcal{R}$.

The current is obtained by making this decomposition in $\mathbf{T}$. Its restriction to inputs coming from $\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})$ is

$$\begin{aligned}\widetilde{\mathbf{T}}\Big|_{\mathcal{A}\mathrm{t}_\Omega(\mathcal{R})} = \frac{1}{2}\int_X \mathrm{c} \wedge e^{\iota_\mu}(1-\iota_\mu P_3)\mathrm{c} + \int_X \beta \wedge D_0\gamma \\ + \frac{1}{2}\int_X \beta \wedge D_2\beta + \frac{1}{2}\int_X \gamma \wedge D_{-2}\gamma.\end{aligned} \tag{4.58}$$

Indeed, the last three terms are precisely the decomposition of $\frac{1}{2}\int_X \Omega(\phi \wedge D\phi)$. For an input α coming from $\Omega^1_X \otimes \mathcal{R}^*$, the nonzero Taylor components are

$$\widetilde{\mathbf{T}}^{(1)}_{\alpha} = \int_X \mathrm{c} \wedge \big((\alpha_1, \beta) + (\alpha_{-1}, \gamma)\big), \tag{4.59}$$

$$\widetilde{\mathbf{T}}^{(2)}_{D;\alpha} = \int_X (\iota_\mu \mathrm{c}) \wedge \big((\alpha_1, \beta) + (\alpha_{-1}, \gamma)\big), \tag{4.60}$$

$$\widetilde{\mathbf{T}}^{(3)}_{D,D';\alpha} = -2\int_X (\iota_\mu \iota_{\mu'} \mathrm{c}) \wedge \big((\alpha_1, \beta) + (\alpha_{-1}, \gamma)\big), \tag{4.61}$$

where $\mu' = \#D_0'$. As before, all Taylor components containing two or more inputs from $\Omega^1_X \otimes \mathcal{R}^*$ vanish. In particular, the arity-one part of the polarized current is

$$\begin{aligned}\widetilde{\mathbf{T}}^{(1)}_{D+\alpha} &= \frac{1}{2}\int_X \mathrm{c} \wedge \iota_\mu \mathrm{c} + \int_X \beta \wedge D_0 \gamma + \frac{1}{2}\int_X \beta \wedge D_2 \beta + \frac{1}{2}\int_X \gamma \wedge D_{-2}\gamma \\ &\quad + \int_X \mathrm{c} \wedge (\alpha_1, \beta) + \int_X \mathrm{c} \wedge (\alpha_{-1}, \gamma). \end{aligned} \tag{4.62}$$

Proposition 4.11. *The Taylor components (4.58)–(4.61) define an L_∞ local current*

$$\widetilde{\mathbf{T}} \colon \widetilde{\mathcal{E}}(3|6) \rightsquigarrow \mathcal{O}_{\mathrm{loc}}(\widetilde{\mathcal{V}})[-1] \tag{4.63}$$

which satisfies the equivariant classical master equation.

Proof. Both $\widetilde{\mathcal{E}}(3|6)$ and $\widetilde{\mathcal{V}}$ are obtained from $\mathcal{E}(3|6)_{\mathcal{R}}$ and $\mathcal{V}$ by the same consistent regrading. This changes only the integer degrees: the differential, the brackets, the Poisson structure, and the formulas for the Taylor components are unchanged. Since the parity of the new degree agrees with the original super parity, the Koszul signs are unchanged as well. Consequently the equivariant master equation for $\widetilde{\mathbf{T}}$ is the regrading, term by term, of the master equation for $\mathbf{T}$ proved in Theorem 4.10. □

For example, a holomorphic vector field $v \in \mathcal{T}_S \subset \widetilde{\mathcal{E}}(3|6)^1$ has only an α_1-component, and its current is

$$\widetilde{\mathbf{T}}^{(1)}_v = \int_{C\times S} \beta \wedge \iota_v \mathrm{c}. \tag{4.64}$$

5 Quantization of symmetries via factorization algebras

We now turn to the factorization algebra of quantum observables of the holomorphic six-dimensional theory, as well as the factorization algebra of currents associated to symmetries of the six-dimensional theory. We recall how a factorization enhancement of Noether's theorem shows that symmetries can be encoded as a map between these factorization algebras.

5.1 Observables of free QFT

In [CG21, §8], the factorization algebra associated to a nondegenerate free BV theory is constructed. In this section, we elaborate this construction to handle the free theories of Poisson type covered by definition 4.1. For other related discussions, see [BY16; GRW22; SW23a].

Recall that the first piece of data which defines a free theory in the BV formalism is an elliptic complex of vector bundles (E, Q) on the spacetime manifold M. To an open set $U \subset M$, we associate a space of fields, which is the space of sections $\mathcal{E}(U) = \Gamma(U, E)$. The classical observables are $\mathcal{O}(\mathcal{E}(U))$, an appropriate algebra of functions on the space of fields $\mathcal{E}(U)$. Since $\mathcal{E} \colon U \subset M \mapsto \mathcal{E}(U)$ is a sheaf on spacetime, the assignment

$$U \mapsto \mathcal{O}(\mathcal{E}(U)) \tag{5.1}$$

is a precosheaf. The multiplication on $\mathcal{O}(\mathcal{E})$ endows this with the structure of a *commutative* factorization algebra [CG17]. This is the factorization algebra of classical observables of a free theory.

For the purposes of this work, the appropriate algebra of functions on $\mathcal{E}(U)$ is defined by

$$\mathcal{O}(\mathcal{E}(U)) = \mathrm{Sym}\left(\Gamma_c(U, E^!)\right). \tag{5.2}$$

This makes the cosheaf property of $U \mapsto \mathcal{O}(\mathcal{E}(U))$ obvious.

In general, the quantization of such factorization algebras to non-commutative factorization algebras can be made rigorous using the Batalin–Vilkovisky (BV) formalism. Classically, in the BV formalism, the space of fields is equipped with a local symplectic form of degree -1. From this data, one constructs a shifted Poisson bracket on the factorization algebra (5.2), endowing it with the structure of a so-called $\mathbb{P}_0$ factorization algebra. The BV Laplacian associated to this shifted Poisson bracket is essentially unique; deforming the differential by the BV Laplacian defines the quantized factorization algebra [CG21, §8].

A Poisson degenerate free BV theory is equipped with a local Poisson bivector of degree $+1$. The construction of a BV Laplacian associated to a general $\mathbb{P}_0$ algebra is very difficult (even more difficult than the quantization of ordinary Poisson algebras). For the class of Poisson structures we study, though, the quantization is no more difficult than in the non-degenerate case. Let (E, Q, π) be the data of a Poisson degenerate free BV theory. Recall that this means that (E, Q) is an elliptic complex of vector bundles on spacetime M; if we view $E^!$ as an abelian local dg Lie algebra, then π can be viewed as a quadratic local cocycle of degree $+1$ on $E^!$. This local cocycle then defines, for each open set $U \subset M$, a central extension of dg Lie algebras

$$0 \to \mathbf{C}[-1] \to \mathcal{H}_c(U) \to \mathcal{E}^!_c(U) \to 0. \tag{5.3}$$

Note that $\mathcal{E}^!_c$ is a *cosheaf* of abelian dg Lie algebras.

The Chevalley–Eilenberg complex of $\mathcal{H}_c(U)$ is of the form

$$C_\bullet(\mathcal{H}_c(U)) = \left(\mathrm{Sym}(\mathcal{E}^!_c(U))[c], Q + c\triangle_\pi\right) \tag{5.4}$$

where $\triangle_\pi$ is the piece of the Chevalley–Eilenberg differential involving π. The variable c, of cohomological degree zero, is dual to the central generator of $\mathcal{H}_c(U)$. The assignment $U \mapsto C_\bullet(\mathcal{H}_c(U))$ is a special example of the *enveloping factorization algebra* of a local Lie algebra [CG17].

Definition 5.1. Let (E, Q, π) be a Poisson degenerate free BV theory. The free BV quantization of the factorization algebra $\mathcal{O}(\mathcal{E})$ is the factorization algebra Obs which assigns to an open set $U \subset M$ the cochain complex

$$\mathrm{Obs}(U) \stackrel{\mathrm{def}}{=} \left(\mathrm{Sym}(\mathcal{E}^!_c(U)), Q + \triangle_\pi\right). \tag{5.5}$$

In other words, it is the specialization of the enveloping factorization algebra $C_\bullet(\mathcal{H}_c)$ obtained by setting $c = 1$.

We note that Obs of course depends on the input data (E, Q, π) of which it is the free BV quantization, but we suppress this from the notation.

The filtration by polynomial degree of an observable induces a filtration of the factorization algebra Obs. Since π is quadratic, there is an isomorphism of commutative factorization algebras

$$\mathrm{gr}\,\mathrm{Obs} \simeq \mathcal{O}(\mathcal{E}). \tag{5.6}$$

In particular, we have the natural "dequantization" map of factorization algebras $\mathrm{Obs} \to \mathcal{O}(\mathcal{E})$.

5.1.1. Although neither the abelian $\mathcal{N} = (2, 0)$ theory nor its twists are Lagrangian theories, they nevertheless fall into the class of Poisson degenerate free BV theories that we have treated here. Thus the free BV quantization of these theories is defined by the methods we have presented. We fix notation for the resulting factorization algebras in the following definition.

Definition 5.2. Let X be a complex three-fold equipped with a rank two holomorphic vector bundle $\mathcal{R}$, and an isomorphism $\Omega\colon \wedge^2 \mathcal{R} \cong K_X$. Define the factorization algebra $\mathrm{Obs}_{\mathcal{R}}$ on X to be the free BV quantization of the holomorphic twist of the six-dimensional abelian $\mathcal{N} = (2, 0)$ theory as recalled in definition 4.2.

Similarly, let $\widetilde{\mathrm{Obs}}$ be the free BV quantization of the polarized holomorphic twist on $C \times S$ as defined in definition 4.3.

Note that the factorization algebra $\mathrm{Obs}_{\mathcal{R}}$ inherits a $\mathbf{Z} \times \mathbf{Z}/2$-grading from the space of fields $\mathcal{V}_{\mathcal{R}}$ underlying the holomorphic theory whereas $\widetilde{\mathrm{Obs}}$ is simply $\mathbf{Z}$-graded.

5.2 Quantum currents from Noether's theorem

We begin by quickly recalling the statement of the quantum Noether theorem for factorization algebras following [CG21]. The input data include a local Lie algebra $\mathcal{L}$, which we think of as describing some symmetry of a classical field theory with space of fields $\mathcal{E}$. At the classical level, we recalled above that such a symmetry is encoded by the data of a local functional

$$J \in \mathcal{O}_{\mathrm{loc}}(\mathcal{L}[1] \oplus \mathcal{E}) \tag{5.7}$$

satisfying the equivariant classical master equation

$$\mathrm{d}_{CE} J + QJ + \frac{1}{2}\{J, J\} = 0, \tag{5.8}$$

where d_{CE} is the Chevalley–Eilenberg differential for $\mathcal{L}$, Q is the BRST operator for $\mathcal{E}$, and $\{-, -\}$ is the BV bracket. In [CG21] it is shown how this data can be used to construct a map of factorization algebras

$$J\colon \mathrm{Sym}(\mathcal{L}_c[1]) \to \mathcal{O}(\mathcal{E}) \tag{5.9}$$

with the property that J preserves the brackets on the source and target. On the source, the bracket is induced from the Lie bracket on $\mathcal{L}$; on the target, the bracket is the BV bracket.

Note that polynomial degree also induces a filtration on the factorization algebra $\mathbf{U}_\Theta(\mathscr{L})$ for any local Lie algebra $\mathscr{L}$. If Θ is at least quadratic, then the associated graded of this filtration is the factorization algebra which assigns to an open set U the cochain cochain complex

$$(\mathrm{Sym}(\mathscr{L}_c(U)[1]), \mathrm{d}_{\mathscr{L}}) \tag{5.10}$$

where $\mathrm{d}_{\mathscr{L}}$ is the *linear* differential on $\mathscr{L}$.

5.2.1. The following is a corollary of the quantum factorization algebra Noether theorem, as formulated in [CG21], in the case of a free quantum field theory.

Theorem 5.3 ([CG21])**.** *Suppose that (E, Q, π) is a free theory in the BV formalism with a classical symmetry $J\colon \mathscr{L} \to \mathcal{O}_{\mathrm{loc}}(\mathscr{E})$ and let* Obs *denote the factorization algebra of quantum observables. There is a class*

$$\Theta_{\mathscr{E}} \in H^1_{loc}(\mathscr{L}) \tag{5.11}$$

together with a filtration preserving map of factorization algebras

$$J^q\colon \mathbf{U}_\Theta(\mathscr{L}) \to \mathrm{Obs} \tag{5.12}$$

whose associated graded is the classical symmetry $\mathrm{gr}\, J^q = J$.

To quantize a theory in the BV formalism one must ensure that the quantum action satisfies the so-called quantum master equation. For free theories this is a triviality; the QME always holds. On the other hand, classically we have seen that the data of a symmetry can be encoded by an equivariant version of the classical master equation (5.8). There is an equivariant version of this equivariant quantum master equation. Heuristically, it is of the form

$$\mathrm{d}_{CE} J^q + Q J^q + \frac{1}{2}\{J^q, J^q\} + \triangle J^q \stackrel{?}{=} 0. \tag{5.13}$$

This is a heuristic since to make sense of it rigorously we must apply renormalization to the quantum Noether current J^q. For free theories, the precise form of this equation always holds when we work modulo functions of $\mathscr{L}$ (more accurately, these are functionals on the shift $\mathscr{L}[1]$). Concretely, $\Theta_{\mathscr{E}}$ is the $\mathscr{L}$-dependent part of this expression:

$$\Theta_{\mathscr{E}} \stackrel{\mathrm{def}}{=} \mathrm{d}_{CE} J^q + Q J^q + \frac{1}{2}\{J^q, J^q\} + \triangle J^q. \tag{5.14}$$

5.2.2. The obstruction can be explicitly described in the following relevant situation. Suppose that $(\mathscr{E}, Q, \pi)$ is a *holomorphic* free field theory (of Poisson degenerate type) on a complex manifold X of dimension n. (For the anomaly we will only look at $X = \mathbf{C}^n$.) This means that the sheaf of complexes $(\mathscr{E}, Q)$ is of the form

$$\left(\Omega^{0,\bullet}(\mathbf{C}^n, \mathcal{V}), Q = \overline{\partial} + Q^{\mathrm{hol}}\right) \tag{5.15}$$

where $\mathcal{V}$ is a graded holomorphic vector bundle on X, Q^{hol} is a holomorphic differential operator from $\mathcal{V}$ to $\mathcal{V}$ of degree +1 and square-zero. The shifted Poisson bivector π is required to arise from a holomorphic differential operator

$$\pi^{\mathrm{hol}}\colon (\mathcal{V}^{\mathrm{hol}})^! \to \mathcal{V}^{\mathrm{hol}}[1-n], \tag{5.16}$$

where $\mathscr{V}^{\mathrm{hol}}$ is the sheaf of holomorphic sections of V and n. This is a generalization of the notion of a non-degenerate holomorphic free field theory given in [Wil20]. Since the data of a holomorphic free field theory only really depends on the holomorphic vector bundle $\mathcal{V}$, the holomorphic operator Q^{hol}, and π^{hol} we will write the data of a holomorphic free theory as a triple $(\mathcal{V}, Q^{\mathrm{hol}}, \pi)$.

Proposition 5.4. *Suppose that $(\mathcal{V}, Q^{\mathrm{hol}}, \pi^{\mathrm{hol}})$ is a holomorphic free field theory on $\mathbf{C}^n$ and suppose that J is a classical symmetry by a holomorphic local Lie algebra $\mathscr{L}$. The local functional $\Theta_{\mathscr{E}} \in \mathrm{C}^\bullet_{\mathrm{loc}}(\mathscr{L})$ of cohomological degree $+1$ can be computed as the $\epsilon, L \to 0$ limit of the wheel with $n+1$ vertices, each labeled by J, and n of whose edges are labeled by the propagator $P^{\pi}_{\epsilon<L}$. The remaining edge is labeled by the heat kernel $K^{\pi}_{\epsilon<L}$.*

As a consequence of theorem 5.3 we obtain the following result about quantizing the classical symmetry of the local Lie algebra $\mathscr{E}(3|6)_{\mathcal{R}}$ on the minimal holomorphic twist of the six-dimensional $\mathcal{N} = (2,0)$ theory. Recall that this theory carries a classical $\mathscr{E}(3|6)_{\mathcal{R}}$ background that is encoded by the classical current $\mathbf{T}$ constructed in section 4.4. As a general consequence of the factorization algebra Noether theorem, we obtain the following.

Proposition 5.5. *Fix a three-fold X with a rank two holomorphic vector bundle $\mathcal{R}$ together with an isomorphism $\wedge^2\mathcal{R} \cong \mathcal{K}_X$. Let*

$$\Theta_{fr} \in H^1_{loc}(\mathscr{E}(3|6)_{\mathcal{R}}) \tag{5.17}$$

be the obstruction for the $E(3|6)$ stress tensor $\mathbf{T}$ of the free theory to solve the equivariant quantum master equation (5.14). There is a map of factorization algebras

$$\mathbf{T}^q \colon \mathbf{U}_{\Theta}(\mathscr{E}(3|6)_{\mathcal{R}}) \to \mathrm{Obs}_{\mathcal{R}} \tag{5.18}$$

such that $\mathrm{gr}\,\mathbf{T}^q = \mathbf{T}$.

Using proposition 5.4 we can describe the obstruction Θ appearing in the proposition above using explicit Feynman diagrams. Let's consider, for example, the symmetry by the sub local Lie algebra

$$\mathfrak{sl}(2) \otimes \Omega^{0,\bullet}_X \subset \mathscr{E}(3|6)_{\mathcal{R}}. \tag{5.19}$$

The anomaly for this symmetry is given by a certain sum of weights of Feynman graphs. From the proposition, we know that the only relevant graph is simply a wheel with four vertices. The resulting weight gives the anomaly; when restricted to the $\mathfrak{sl}(2)$-valued fields, its value is

$$\Theta|_{\mathfrak{sl}(2)} = \frac{1}{(2\pi \mathrm{i})^3} \int_X \mathrm{Tr}(A\partial A)\,\mathrm{Tr}(\partial A \partial A). \tag{5.20}$$

The subfactorization algebra

$$\mathbf{U}_{\Theta|_{\mathfrak{sl}(2)}}(\mathfrak{sl}(2) \otimes \Omega^{0,\bullet}_X) \subset \mathbf{U}_{\Theta}(\mathscr{E}_{(3|6)}) \tag{5.21}$$

has appeared in [GW18] as a higher-dimensional generalization of the Kac-Moody factorization/vertex algebra. In particular, on $X = \mathbf{C}^3 - \{0\}$ it is shown that the factorization algebra encodes a higher Kac–Moody L_∞ algebra as defined in [FHK19]. In this dimension, higher Kac–Moody algebras are classified by quartic invariant polynomials on the Lie algebra. The particular higher Kac–Moody algebra is an L_∞ central extension of the form

$$0 \to \mathbf{C} \to \widehat{\mathfrak{sl}(2)}_{3,\theta} \to \mathfrak{sl}(2) \otimes \mathrm{A}_3 \to 0 \tag{5.22}$$

where A_3 is the Jouanolou model for the derived global sections of the structure sheaf on $\mathbf{C}^3 - \{0\}$ and where θ is proportional to the quartic polynomial $X \mapsto \mathrm{Tr}(X^4)$.

5.2.3. Similarly, we have the following polarized twisted version of the above proposition.

Proposition 5.6. *Fix a complex manifold X of dimension 3 and let*

$$\widetilde{\Theta}_{fr} \in H^1_{loc}(\widetilde{\mathscr{E}}(3|6)) \tag{5.23}$$

be the obstruction for $\widetilde{\mathbf{T}}$, see equation (4.62), to solve the equivariant quantum master equation (5.14). There is a map of factorization algebras

$$\widetilde{\mathbf{T}}^q : \mathbf{U}_{\widetilde{\Theta}}(\widetilde{\mathscr{E}}_{(3|6)}) \to \widetilde{\mathrm{Obs}}. \tag{5.24}$$

such that $\mathrm{gr}\, \widetilde{\mathbf{T}}^q = \widetilde{\mathbf{T}}$.

5.3 The anomaly from Riemann–Roch

In this section we explicitly describe the anomalies defined above in terms of characteristic classes and relate them to well-known expressions for fivebrane anomalies. The quantum anomaly class Θ is a local cocycle of $\mathscr{E}_{(3|6)}$ of cohomological degree +1. On $X = \mathbf{C}^3$ we can use proposition 5.4 to compute this anomaly explicitly. We can appeal to the Grothendieck–Riemann–Roch theorem to geometrically interpret this anomaly. To set this up, consider the even sub local Lie algebra

$$\mathscr{E}^{\mathrm{even}}_{(3|6)} = \Omega^{0,\bullet}(X, \mathrm{T}_X) \ltimes \Omega^{0,\bullet}(X, \mathfrak{sl}(2)), \tag{5.25}$$

and the induced cochain map

$$\mathrm{C}^\bullet_{\mathrm{loc}}(\mathscr{E}_{(3|6)}) \to \mathrm{C}^\bullet_{\mathrm{loc}}(\mathscr{E}^{\mathrm{even}}_{(3|6)}). \tag{5.26}$$

In other words, a local cocycle for $\mathscr{E}_{(3|6)}$ can always be viewed as a local cocycle for its even subalgebra.

The utility of this homomorphism is that the local cohomology of $\mathscr{E}^{\mathrm{even}}_{(3|6)}$ is known [Kho06; Wil24b] and can be identified with certain universal characteristic classes.

On $X = \mathbf{C}^3$, one has an isomorphism

$$H^1_{loc}(\mathscr{E}^{\mathrm{even}}_{3|6}(\mathbf{C}^3)) \simeq H^8(BU(3) \times BSU(2)). \tag{5.27}$$

This cohomology has a description in terms of universal characteristic classes. Let $\mathrm{ch}_i \in H^{2i}(BU(3))$, $i = 1, 2, 3$ be the universal Chern characters and let $\kappa \in H^4(BSU(2))$ denote the generator. Then, the anomaly we find will be given by polynomial expressions of $\mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3, \kappa$ of total degree 8 where ch_i carries degree $2i$ and κ carries degree four.

The following result can be found in [Wil24a].

Proposition 5.7. *Let (V, Q, π) be a holomorphic free field theory on $X = \mathbf{C}^n$. Suppose that V is equipped with a G-action preserving π and consider the resulting $\mathcal{T}_X \ltimes \Omega^{0,\bullet}_X \otimes \mathfrak{g}$-background.*

The anomaly to lifting this to an inner quantum $\mathcal{T}_X \ltimes \Omega^{0,\bullet}_X \otimes \mathfrak{g}$-background is given by the class

$$\frac{1}{2}\,\mathrm{Td}\cdot \mathrm{ch}_{U(n)\times G}(V)\big|_{2n+2} \in H^{2n+2}(BU(n)\times BG) \cong H^1_{loc}(\mathcal{T}(\mathbf{C}^n) \ltimes \Omega^{0,\bullet}(\mathbf{C}^n)\otimes \mathfrak{g}), \tag{5.28}$$

where $\mathrm{ch}_{U(n)\times G}(V)$ is the $U(n)\times G$-character of the representation V.

This anomaly is a shadow of the mixed gravitational and R-symmetry/flavor anomaly familiar in supersymmetric field theories. For this reason we will refer to it as a mixed holomorphic-G anomaly.

Let $\Theta_{3|6}$ denote the anomaly to quantizing the classical $\mathcal{E}_{(3|6)}$ background on the (minimal) holomorphic twist of the $\mathcal{N} = (2,0)$ theory given by the local current $\mathbf{T}_{3|6}$ of equation (??). Using proposition 5.7 in the case $G = SU(2)$, we compute the restriction of this anomaly to the subalgebra $\mathcal{E}^{\mathrm{even}}_{(3|6)}$ There are two pieces to this multiplet: the holomorphic twist of the $\mathcal{N} = (1,0)$ tensor multiplet and the holomorphic twist of the $\mathcal{N} = (1,0)$ hypermultiplet.

Accordingly, the anomaly splits as

$$\Theta_{3|6} = \Theta^{\mathrm{tensor}}_{3|6} + \Theta^{\mathrm{hyper}}_{3|6}. \tag{5.29}$$

In the next two sections we will characterize the mixed holomorphic-$SU(2)$ anomaly of the holomorphic twist of the superconformal theory.

The graded holomorphic vector bundle underlying the twisted tensor multiplet is

$$V = \wedge^2 \mathrm{T}^*[1] \oplus K. \tag{5.30}$$

Its universal Chern character is

$$\mathrm{ch}(V) = 1 - \mathrm{ch}^* = -2 + \mathrm{ch}_1 - \mathrm{ch}_2 + \mathrm{ch}_3 - \mathrm{ch}_4\,. \tag{5.31}$$

where ch_i is the ith Chern character associated to the universal tangent bundle. Since ch_4 is an expression in c_1, c_2, c_3 we can express it in terms of the lower Chern characters as

$$\mathrm{ch}_4 = \frac{1}{6}\left(\frac{1}{24}\,\mathrm{ch}_1^4 - \frac{1}{2}\,\mathrm{ch}_1^2\,\mathrm{ch}_2 + 2\,\mathrm{ch}_1\,\mathrm{ch}_3 + \frac{1}{2}\,\mathrm{ch}_2^2\right), \tag{5.32}$$

Also we record the following identity

$$\mathrm{Td}_4 = \frac{1}{120}\left(\frac{23}{72}\,\mathrm{ch}_1^4 - \frac{4}{3}\,\mathrm{ch}_1^2\,\mathrm{ch}_2 + \frac{1}{3}\,\mathrm{ch}_1\,\mathrm{ch}_3 + \frac{1}{2}\,\mathrm{ch}_2^2\right). \tag{5.33}$$

Expanding the five terms appearing in the degree 8 part of

$$\frac{1}{2}\operatorname{Td}(X)\operatorname{ch}(V), \tag{5.34}$$

we see that the anomaly is given by

$$\Theta^{\text{tensor}}_{3|6}\Big|_{\text{even}} = \frac{1}{2\cdot 4!}\left(\frac{37}{180}\operatorname{ch}_1^4 - \frac{22}{15}\operatorname{ch}_1^2\operatorname{ch}_2 + \frac{58}{15}\operatorname{ch}_1\operatorname{ch}_3 - \frac{1}{5}\operatorname{ch}_2^2\right). \tag{5.35}$$

The super $\mathbf{Z}$-graded holomorphic vector bundle underlying the holomorphic twist of the hypermultiplet is $W = \Pi K^{1/2}\otimes \mathbf{C}^2[1]$.

We expand the degree 8 part of

$$\frac{1}{2}\operatorname{Td}\operatorname{ch}_{U(3)\times SU(2)}(W) \tag{5.36}$$

and find the result

$$\Theta^{\text{hyper}}_{3|6}\Big|_{\text{even}} = \frac{1}{4!}\left(\frac{1}{720}\operatorname{ch}_1^4 - \frac{1}{60}\operatorname{ch}_1^2\operatorname{ch}_2 + \frac{1}{10}\operatorname{ch}_2^2 + \frac{1}{15}\operatorname{ch}_1\operatorname{ch}_3 - \operatorname{ch}_2\kappa + \kappa^2\right). \tag{5.37}$$

We conclude that the total even part to the $\mathscr{E}_{(3|6)}$ anomaly for the free theory is

$$\begin{aligned}\Theta_{3|6}\big|_{\text{even}} &= \Theta^{\text{tensor}}_{3|6}\Big|_{\text{even}} + \Theta^{\text{hyper}}_{3|6}\Big|_{\text{even}} \\ &= \frac{1}{4!}\left(\frac{5}{48}\operatorname{ch}_1^4 - \frac{3}{4}\operatorname{ch}_1^2\operatorname{ch}_2 + 2\operatorname{ch}_1\operatorname{ch}_3 - \operatorname{ch}_2\kappa + \kappa^2\right).\end{aligned} \tag{5.38}$$

Compare this with the expression of the anomaly computed in the (untwisted) supersymmetric theory [Wit97]. The anomaly polynomial for the supersymmetric $\mathcal{N} = 1$ tensor multiplet is of the form

$$\Theta^{\text{phys}}_{\text{tensor}}(p_1, p_2, c_2, \kappa) \tag{5.39}$$

and depends on the universal Pontryagin classes p_1, p_2, and also an $SU(2)_R$-symmetry part, whose generator we denote c_2. To reduce to the twisted anomaly we set $p_1 \mapsto 2\operatorname{ch}_2$, $p_2 \mapsto 2(\operatorname{ch}_2^2 - 6\operatorname{ch}_4)$, and $c_2 \mapsto -\frac{1}{4}\operatorname{ch}_1^2$ in the physical anomaly, which agrees with what we have computed directly from Riemann–Roch. The latter assignment is fixed by our choice of twisting homomorphism. Upon this specialization, the supersymmetric anomaly (5.39) matches exactly with what we have computed in (5.37).

5.4 The spherical currents

In this section we seek to make the role of $E(3|6)$ as a symmetry more apparent. Recall that in a chiral CFT, conformal symmetry manifests as an explicit representation of the Virasoro Lie algebra at some central charge. The Virasoro Lie algebra is precisely a central extension of the S^1-modes of the local Lie algebra of holomorphic vector fields. In the final part of this section we extract the explicit S^5-supported currents from the local Lie algebra $\mathscr{E}(3|6)$. The content of this section is similar that of [GW18] where the higher Kac–Moody Lie algebra of [FHK19] is recovered from the higher Kac–Moody factorization algebra defined on $\mathbf{C}^d - \{0\}$.

So, consider the factorization algebra $\mathscr{G} = \mathbf{U}_\Theta(\mathscr{E}_{(3|6)})$ defined on punctured affine space $\mathbf{C}^3 - \{0\}$. We consider the "radial quantization" of this factorization algebra, that is, the pushforward along the radial projection map

$$r\colon \mathbf{C}^3 - \{0\} \to \mathbf{R}_{>0}. \tag{5.40}$$

The pushforward factorization algebra $r_*\mathscr{G}$ is not locally constant on $\mathbf{R}_{>0}$ but it admits a dense subfactorization algebra $\mathscr{A}$ that is locally constant. To an interval $I \subset \mathbf{R}_{>0}$ this factorization algebra $\mathscr{A}$ assigns the cochain complex

$$\mathscr{A}(I) = \oplus_{n\in\frac{1}{2}\mathbf{Z}}(\mathscr{G}(r^{-1}I))^{(n)} \subset \mathscr{G}(r^{-1}I) \tag{5.41}$$

where the superscript denotes the pure weight n eigenspace for the action by rescalings $(z_1, z_2, z_3) \mapsto (\lambda z_1, \lambda z_2, \lambda z_3)$.

The one-dimensional factorization algebra $\mathscr{A}$ is locally constant, and hence is equivalent to the data of an explicit A_∞ algebra A. This A_∞ superalgebra is the enveloping algebra of an L_∞ algebra

$$A \simeq U(\mathrm{Vir}_{(3|6)}). \tag{5.42}$$

And this L_∞ algebra is obtained as a central extension

$$0 \to \mathbf{C} \to \mathrm{Vir}_{(3|6)} \to \mathrm{Witt}_{(3|6)} \to 0 \tag{5.43}$$

where $\mathrm{Witt}_{(3|6)}$ is the so-called Jouanolou model for the dg Lie superalgebra $\mathscr{E}_{(3|6)}(\mathbf{C}^3 - \{0\})$ [GWW26; FHK19]. We refer to it as a super "Witt" algebra as it is a super extension of the Lie algebra of derived sections of the tangent sheaf on punctured affine space. Likewise, the centrally extended L_∞ algebra is akin to the Virasoro algebra in CFT.

The L_∞ superalgebra $\mathrm{Vir}_{(3|6)}$ is concentrated in cohomological degrees $0, 1, 2$. It contains, as the nonnegative scaling modes, the Lie superalgebra $E(3|6) \subset \mathrm{Vir}_{(3|6)}$. We point out that that the central extension cocycle vanishes identically on these non-negative modes.

So far, we hope the situation outlined in this section reminds the reader of the Witt and Virasoro algebras of ordinary chiral CFT. Furthermore, as a consequence of proposition 5.5 we can build representations of $\mathrm{Vir}_{(3|6)}$ from $E(3|6)$-structured factorization algebras. Indeed, suppose that $\mathscr{F}$ is such a factorization algebra. Then 5.5 supplies the cochain complex

$$V_{\mathscr{F}} \overset{\mathrm{def}}{=} \oplus_{n\in\frac{1}{2}\mathbf{Z}_+}\mathscr{F}(D)^{(n)} \tag{5.44}$$

where D is any 3-disk centered at the origin, with the structure of a $\mathrm{Vir}_{(3|6)}$-representation. The complex $V_{\mathscr{F}}$ is the analog of the "state space" of a chiral CFT.

6 Holomorphic localization

We study the deformations of $E(3|6)$ structures on the product threefold $C \times S$ where C is a complex curve and S is a complex surface. Such deformations are determined by Maurer–Cartan elements for the local Lie algebra $\widetilde{\mathcal{E}}(3|6)$; though we consider just those that are holomorphic for the fixed complex structure on $C \times S$.

We will find that a particular class of deformation is determined by holomorphic vector field $v \in \Gamma(\mathcal{T}_S)$ on the complex surface. Such a deformation also deforms the factorization algebras built from the $E(3|6)$ structure; such as the enveloping factorization algebra of $\widetilde{\mathcal{E}}(3|6)$ and also the factorization algebra of observables $\widetilde{\mathrm{Obs}}$ underlying the holomorphic twist of the $\mathcal{N} = (2,0)$ theory.

In the case that $S = \mathbf{C}^2$ and $v = \mathsf{Eu}$ is the Euler vector field on $\mathbf{C}^2$, let $\pi\colon C \times \mathbf{C}^2 \to C$ be the projection. We summarize the deformation and pushforward determined by this data as a functor

$$\mathrm{Loc}_{\pi,\mathsf{Eu}}\colon \mathrm{Fact}^{(3|6)}(C \times \mathbf{C}^2) \longrightarrow \mathrm{Fact}^{\mathrm{conf}}(C). \tag{6.1}$$

Its source is the category of factorization algebras on $C \times \mathbf{C}^2$ with symmetry by the local Lie algebra $\widetilde{\mathcal{E}}(3|6)$, and its target is the category of chiral conformal factorization algebras on C. This functor is an avatar of the superconformal localization considered in [BRR15].

More generally, given any holomorphic vector field v on any complex surface S we will construct a factorization algebra on $C \times Z_v$ where $Z_v \subset S$ is the zero locus of the vector field v.

6.1 Further twists

Let C be a Riemann surface and S be a complex surface as in 2.1.4. The product threefold $X = C \times S$ admits an $E(3|6)$ structure by taking the rank-two bundle

$$\mathcal{R} = \pi_C^* \Omega^1_C \oplus \pi_S^* \Omega^2_S, \tag{6.2}$$

where $\pi_{C,S}$ are the obvious projections. The consistently $\mathbf{Z}$-graded local Lie algebra $\widetilde{\mathcal{E}}(3|6)$ makes sense in this setting; effectively regrading $\mathcal{R}$ by placing Ω^2_S in degree $+1$ and Ω^1_C in degree -1. Upon regrading, the odd part of $E(3|6)$ becomes smeared out in degrees -1 and $+1$:

$$\Pi\left(\mathcal{T}_X^* \otimes \mathcal{R}^*\right) \rightsquigarrow \left(\mathcal{O}_X \oplus \mathcal{T}_C \boxtimes \Omega^1_S\right)[1] \oplus \left(\mathcal{O}_C \boxtimes \mathcal{T}_S \oplus \Omega^1_C \boxtimes \wedge^2 \mathcal{T}_S\right)[-1] \tag{6.3}$$

6.1.1. We classify holomorphic Maurer–Cartan elements of the exceptional holomorphic Lie algebra on $C \times S$. By such, we mean global holomorphic sections α of $\widetilde{\mathcal{E}}(3|6)$ of cohomological degree $+1$ which satisfy the Maurer–Cartan equation

$$[\alpha, \alpha] = 0.^{4} \tag{6.4}$$

[4] More generally, one could study Maurer–Cartan elements of the *local* Lie algebra $\mathcal{E}(3|6)_{\mathcal{R}}$ given by the Dolbeault complex on $\mathcal{E}(3|6)_{\mathcal{R}}$. Such Maurer–Cartan elements are no longer holomorphic. In part, they deform the complex structure on $C \times S$. We consider only deformations which leave the underlying complex structure unchanged.

Proposition 6.1. *Holomorphic Maurer–Cartan elements of $\widetilde{\mathcal{E}}(3|6)$ are in bijective correspondence with pairs of sections*

$$(V, \lambda) \in \Gamma(\mathcal{O}_C \boxtimes \mathcal{T}_S \oplus \Omega^1_C \boxtimes \wedge^2 \mathcal{T}_S) \tag{6.5}$$

satisfying the equation

$$V \wedge \partial_C V = L_V \lambda \tag{6.6}$$

where ∂_C is the holomorphic de Rham operator along C and L_V denotes the Lie derivative along the vector field V.

Proof. The equation follows from the form of the odd-odd bracket in $\mathcal{E}(3|6)_{\mathcal{R}}$. Only single-derivative terms are relevant, and the $[\lambda, \lambda]$ bracket is identically zero. □

6.1.2. Twists as integrable distributions. The primary type of further twist / Maurer–Cartan element that we study in the remainder of this paper corresponds to the special case that $\lambda = 0$, so that the equation simply becomes

$$V \partial_C V = 0, \quad V \in \Gamma(\mathcal{O}_C \boxtimes \mathcal{T}_S). \tag{6.7}$$

The zero locus of a vertical vector field $V \in \Gamma(\mathcal{O}_C \boxtimes \mathcal{T}_S)$ is a subset

$$Z_V \overset{\text{def}}{=} \{(z, w) \in C \times S \mid V_{(z,w)} = 0\}. \tag{6.8}$$

Let $U_V = (C \times S) \setminus Z_V$ be its complement. On U_V there is a holomorphic line bundle $\mathcal{L}_V$ whose fiber over (z, w) is $\text{span}\{V_{(z,w)}\}$. It is naturally a line subbundle of $\mathcal{O}_C \boxtimes \mathcal{T}_S$. Define the following rank two holomorphic distribution

$$\mathcal{D}_V \overset{\text{def}}{=} \mathcal{T}_C \boxtimes \mathcal{O}_S \oplus \mathcal{L}_V \subset \mathcal{T}_{C \times S}. \tag{6.9}$$

on $C \times S$. The equation (6.7) is precisely the condition that this rank two distribution is integrable.

Conversely, given a rank two integrable distribution $\mathcal{D} \subset \mathcal{T}_{C \times S}$ which contains, as a subbundle, the line bundle $\mathcal{T}_C \boxtimes \mathcal{O}_S$, we can recover this Maurer–Cartan element up to rescaling by a global holomorphic function on C.

6.1.3. Twists by holomorphic vector fields. We consider now the even more specialized case where V is of the form

$$V = 1 \boxtimes v \in \Gamma(\mathcal{O}_C \boxtimes \mathcal{T}_S), \tag{6.10}$$

so that (6.6) is automatically satisfied (we still take $\lambda = 0$). In our applications at the end of this subsection, we will assume, moreover, that the critical locus

$$Z_v = \{w \in S \mid v_w = 0\} \subset S \tag{6.11}$$

is a smooth complex submanifold.

6.1.4. The Ω-background deformation. Maurer–Cartan elements of $\widetilde{\mathcal{E}}(3|6)$ define deformations, or twists, of any object which has a symmetry by $\widetilde{\mathcal{E}}(3|6)$. In the remainder of the paper we characterize such twists in the following special cases of the Maurer–Cartan elements from 6.1.3.

The most important special case is $S = \mathbf{C}^2$, with coordinates w_1, w_2, and

$$v = \mathsf{Eu} \stackrel{\text{def}}{=} w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}. \tag{6.12}$$

Remark 6.2. More generally, the Ω-background considered by [**other Omega background papers?**; BRR15] pertains, at the level of the holomorphic twist, to the family of Maurer–Cartan elements

$$v_{\varepsilon_1,\varepsilon_2} = \varepsilon_1 w_1 \frac{\partial}{\partial w_1} + \varepsilon_2 w_2 \frac{\partial}{\partial w_2}. \tag{6.13}$$

where $\varepsilon_1, \varepsilon_2$ are the rotation parameters in the two factors $\mathbf{C}^2 = \mathbf{C}_{w_1} \times \mathbf{C}_{w_2}$. In this paper, we only consider the localization of the holomorphic twist of the free superconformal theory which is essentially independent of $\varepsilon_1, \varepsilon_2$. Indeed, the localized vertex algebra and symmetry is independent of $\varepsilon_1, \varepsilon_2$ so long as both are nonzero. The localization of the holomorphic twist of the higher rank superconformal theories does depend on these parameters. We consider this in future work.

6.1.5. The half-Ω-background deformation. Let Σ be a different complex curve and set $S = \mathrm{T}^*\Sigma$. Denote by θ the Liouville one-form on $\mathrm{T}^*\Sigma$. Then $\mathrm{d}\theta$ is the standard symplectic form on S. Let v be the unique holomorphic vector field on $\mathrm{T}^*\Sigma$ satisfying

$$\iota_v \mathrm{d}\theta = \theta.$$

Locally, then, v is the Euler vector field along the fibers of the cotangent bundle. For $\Sigma = \mathbf{C}$, this Maurer–Cartan element is locally a degenerate limit of those introduced in Remark 6.2.

6.2 Localization of the free theory

Recall the polarized free theory $\widetilde{\mathscr{V}}$ from definition 4.3, together with its local $\widetilde{\mathcal{E}}(3|6)$ current $\widetilde{\mathbf{T}}$. Fix a holomorphic vector field v on S, regarded as the Maurer–Cartan element of Paragraph 6.1.3. Evaluating the current on this background introduces the quadratic functional

$$I_v = \int_{C\times S} \beta \wedge \iota_v \mathsf{c} \tag{6.14}$$

to the free action. Its Hamiltonian vector field $D_v = \{I_v, -\}$ deforms the linear BRST differential to

$$\bar\partial + Q_v = \bar\partial + Q + D_v. \tag{6.15}$$

Since v satisfies the Maurer–Cartan equation and $\mathbf{T}$ satisfies the equivariant classical master equation, $(\bar\partial + Q_v)^2 = 0$.

Write

$$\mathsf{c} = \mathsf{c}_{2,0} + \mathsf{c}_{1,1}, \qquad \mathsf{c}_{i,j} \in \Omega^i_C \boxtimes \Omega^j_S,$$

$$
\begin{tikzcd}
\underline{-2} & \underline{-1} & \underline{0} \\
 & \Omega^2_{C\times S} \arrow[r, "\partial"] \arrow[dr, dotted, "\iota_v"] & \Omega^3_{C\times S} \\
\mathcal{O}_C \boxtimes \Omega^2_S \arrow[ur, dotted, "\partial\iota_v"] & & \Omega^1_C \boxtimes \mathcal{O}_S.
\end{tikzcd}
$$

Figure 1: The complex $\mathcal{V}^v$ underlying the v-deformation of the polarized free theory. The dotted arrows are the components of D_v.

and write $\mathrm{c}_3 \in \Omega^1_C \boxtimes \Omega^2_S$ for the degree-zero three-form field. The two new components of the holomorphic differential are

$$D_v(\beta) = \partial\iota_v\beta, \qquad D_v(\mathrm{c}_{1,1}) = \iota_v \mathrm{c}_{1,1};$$

all other components vanish. Thus the deformed fields are the Dolbeault resolution of the complex $\mathcal{V}^v$ in Figure 1. We denote this Dolbeault complex by $\mathscr{V}^v$.

The localized model is the following Koszul complex on $C \times S$:

$$
(\mathcal{K}_{\Omega^1}, \mathrm{d} = 1 \boxtimes \iota_v) \overset{\text{def}}{=} \qquad
\begin{array}{ccccc}
\underline{-2} & & \underline{-1} & & \underline{0} \\
\Omega^1_C \boxtimes \Omega^2_S & \xrightarrow{1\boxtimes\iota_v} & \Omega^1_C \boxtimes \Omega^1_S & \xrightarrow{1\boxtimes\iota_v} & \Omega^1_C \boxtimes \mathcal{O}_S.
\end{array}
\tag{6.16}
$$

Equivalently,

$$\mathcal{K}_{\Omega^1} = \left(\Omega^1_C \boxtimes \Omega^{-\bullet}_S, 1 \boxtimes \iota_v\right).$$

Proposition 6.3. *For every holomorphic vector field v on S, there is a cochain map of complexes of sheaves on $C \times S$*

$$\Phi_v \colon \mathcal{V}^v \longrightarrow \mathcal{K}_{\Omega^1} \tag{6.17}$$

defined on homogeneous sections by

$$
\begin{aligned}
&\Phi_v(\beta) = \partial_C\beta, \qquad && \Phi_v(\mathrm{c}_{2,0}) = 0, \qquad && \Phi_v(\mathrm{c}_{1,1}) = \mathrm{c}_{1,1}, \\
&\Phi_v(\gamma) = \gamma, && \Phi_v(\mathrm{c}_3) = 0.
\end{aligned}
\tag{6.18}
$$

Here β has degree -2; $\mathrm{c}_{2,0}$ and $\mathrm{c}_{1,1}$ have degree -1; and c_3 and $\gamma \in \Omega^1_C \boxtimes \mathcal{O}_S$ have degree zero.

Proof. The only nontrivial square beginning with β commutes because

$$(1 \boxtimes \iota_v)\partial_C\beta = \partial_C\iota_v\beta. \tag{6.19}$$

The right-hand side is precisely the $\Omega^1_C \boxtimes \Omega^1_S$ component of $\partial\iota_v\beta$. On $\mathrm{c}_{1,1}$, both differentials give $\iota_v\mathrm{c}_{1,1}$; the $\partial\mathrm{c}$ component lands in c_3 and is killed by Φ_v. The components $\mathrm{c}_{2,0}$ and c_3 are killed on both sides, while Φ_v is the identity on γ. Hence Φ_v is a cochain map. □

The map Φ_v need not be a quasi-isomorphism for an arbitrary vector field. It is one in each of the two cases used below.

Lemma 6.4. *Let $S = \mathbf{C}^2$ and let*

$$v = \mathsf{Eu} = w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}.$$

Then Φ_{Eu} is a quasi-isomorphism of complexes of sheaves on $C \times \mathbf{C}^2$.

Proof. It is enough to show that the mapping cone is acyclic on stalks. Away from $C \times \{0\}$, choose a holomorphic one-form η with $\eta(v) = 1$. Exterior multiplication by η contracts the relevant Koszul complexes, and hence the mapping cone.

Near $C \times \{0\}$, filter the mapping cone by holomorphic form degree along C. On the associated graded, the identity components in (6.18) cancel and the remaining two-term complex has differential

$$L_{\mathsf{Eu}} = [\partial_S, \iota_{\mathsf{Eu}}] \colon \Omega^2_{\mathbf{C}^2} \longrightarrow \Omega^2_{\mathbf{C}^2}.$$

On monomial top forms,

$$L_{\mathsf{Eu}} \left(w_1^m w_2^n \, \mathrm{d}w_1 \wedge \mathrm{d}w_2 \right) = (m + n + 2) w_1^m w_2^n \, \mathrm{d}w_1 \wedge \mathrm{d}w_2.$$

This operator is invertible on the stalk at the origin; its inverse preserves convergent power series. The associated graded cone, and hence the cone itself, is therefore acyclic. □

Lemma 6.5. *Let $S = \mathrm{T}^*\Sigma$ and let v be the fiberwise Euler vector field from Paragraph 6.1.5. Then Φ_v is a quasi-isomorphism of complexes of sheaves on $C \times \mathrm{T}^*\Sigma$.*

Proof. Away from the zero section, the Koszul contraction from the preceding proof applies. Near the zero section, choose a coordinate z on Σ and the induced fiber coordinate p, so that $v = p \frac{\partial}{\partial p}$. The associated-graded mapping cone is again controlled by the Lie derivative on holomorphic top forms, and

$$L_v \left(f(z) p^n \, \mathrm{d}z \wedge \mathrm{d}p \right) = (n + 1) f(z) p^n \, \mathrm{d}z \wedge \mathrm{d}p.$$

Thus L_v is invertible on every stalk along the zero section. The same filtered-cone argument proves that Φ_v is a quasi-isomorphism. □

6.2.1. Compatibility with the shifted Poisson structures. Applying Dolbeault forms to (6.17), and retaining the notation Φ_v, gives a map $\mathscr{V}^v \to \mathscr{K}_{\Omega^1}$, where

$$\mathscr{K}_{\Omega^1} = \Omega^{0,\bullet}(C \times S, \mathcal{K}_{\Omega^1}).$$

Using the wedge pairing on the S-forms to identify the density-valued dual, $\mathscr{K}_{\Omega^1}$ has Poisson operator

$$\pi_{\mathscr{K}} = \mathbb{1}_{\Omega_S^{-\bullet,\bullet}} \boxtimes \partial_C \colon \Omega_S^{-\bullet,\bullet} \boxtimes \Omega_C^{0,\bullet} \longrightarrow \Omega_S^{-\bullet,\bullet} \boxtimes \Omega_C^{1,\bullet}. \tag{6.20}$$

It commutes with $\overline{\partial} + 1 \boxtimes \iota_v$, so it defines a Poisson-degenerate free BV theory.

Proposition 6.6. *For every holomorphic vector field v on S, the map Φ_v preserves the (-1)-shifted Poisson structures. More precisely, if*

$$\Phi_v^! \colon \mathcal{K}_{\Omega^1}^! \longrightarrow (\mathcal{V}^v)^!$$

is the density-valued dual of Φ_v, then

$$\int_{C\times S} \langle \Phi_v(e), \lambda\rangle = \pm \int_{C\times S} \langle e, \Phi_v^!(\lambda)\rangle,$$

with the sign determined by the total degrees, and

$$\Phi_v \circ \pi_{\mathcal{V}^v} \circ \Phi_v^! = \pi_{\mathcal{K}}. \tag{6.21}$$

Consequently, in the situations of Lemmas 6.4 and 6.5, Φ_v is an equivalence of classical Poisson-degenerate free BV theories on $C \times S$.

Proof. The source Poisson structure is the direct sum of the $\partial_{C\times S}$ operator on the c-fields and the Serre-duality pairing between β and γ. On the latter summand, Φ_v is ∂_C on β and the identity on γ. Its pushforward is therefore the ∂_C operator between the Ω_S^2 and Ω_S^0 components of $\mathcal{K}_{\Omega^1}$.

On the c-summand, only $\mathrm{c}_{1,1} \in \Omega_C^1 \boxtimes \Omega_S^1$ survives. Projecting $\partial_{C\times S} = \partial_C + \partial_S$ to this component kills ∂_S and leaves ∂_C, which is the Ω_S^1 component of (6.20). These are all three S-form degrees of $\pi_{\mathcal{K}}$, and there are no mixed terms because the source Poisson structure is block diagonal. This proves (6.21); the final assertion follows from the preceding two lemmas. ◻

6.2.2. Interpretation as a derived zero locus. For a vector bundle $\mathcal{E} \to X$ and a section $s \in \Gamma(X, \mathcal{E})$, let $\mathbf{Z}(s)$ denote its derived zero locus and $i_s \colon \mathbf{Z}(s) \to X$ its canonical map:

$$\begin{tikzcd} \mathbf{Z}(s) \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{hPB}"] & X \arrow[d, "0"] \\ X \arrow[r, "s"] & \mathrm{Tot}(\mathcal{E}). \end{tikzcd} \tag{6.22}$$

As a complex of $\mathcal{O}_X$-modules, the pushforward of its structure sheaf has the Koszul model

$$i_{s,*}\mathcal{O}_{\mathbf{Z}(s)} \simeq (\wedge^{-\bullet}\mathcal{E}^*, \iota_s)\,. \tag{6.23}$$

For $\mathcal{E} = \mathcal{T}_S$ and $s = v$, write $\mathbf{Z}_v = \mathbf{Z}(v)$ and let $j_v \colon C \times \mathbf{Z}_v \to C \times S$ be the induced derived closed immersion. Equation (6.23) gives a canonical identification

$$\mathcal{K}_{\Omega^1} \simeq j_{v,*}\left(\Omega_C^1 \boxtimes \mathcal{O}_{\mathbf{Z}_v}\right). \tag{6.24}$$

Together with the preceding results, this makes the two localizations completely explicit.

6.2.3. Localization to C. The projection $\pi \colon C \times S \to C$ and the vector field v give a Poisson BV theory on C by direct image of the Koszul model. On an open set $U \subset C$, its fields are

$$(\pi_*\mathcal{K}_{\Omega^1})(U) = \Omega^{0,\bullet}\left(U \times S, \Omega_U^1 \boxtimes \Omega_S^{-\bullet}\right), \qquad \mathrm{d} = \bar{\partial} + \iota_v. \tag{6.25}$$

Equivalently, if

$$\mathcal{A}_v = \left(\Omega_S^{-\bullet,\bullet}, \overline{\partial} + \iota_v\right) \simeq \mathbf{R}\Gamma(\mathbf{Z}_v, \mathcal{O}_{\mathbf{Z}_v}),$$

then the fields are $\Omega_C^{1,\bullet}\widehat{\otimes}\mathcal{A}_v$. The Poisson operator is induced from (6.20); using the Koszul–Serre pairing on A_v, it is $\partial_C \otimes \mathbb{1}_{\mathcal{A}_v}$. This is the underlying free Poisson BV construction; its formulation in terms of pushforward of factorization algebras will be given below.

For $S = \mathbf{C}^2$ and $v = \mathsf{Eu}$, the components (w_1, w_2) form a regular sequence, so $\mathbf{Z}_v$ is the ordinary point $\{0\}$. If $i_{\mathsf{Eu}} \colon C \times \{0\} \hookrightarrow C \times \mathbf{C}^2$, then there is a sheafwise equivalence

$$\mathcal{V}^{\mathsf{Eu}} \simeq i_{\mathsf{Eu},*}\Omega_C^1. \tag{6.26}$$

In particular, for the projection $\pi \colon C \times \mathbf{C}^2 \longrightarrow C$,

$$\pi_* \mathcal{V}^{\mathsf{Eu}} \simeq \Omega_C^1.$$

Since $\mathcal{A}_{\mathsf{Eu}} \simeq \mathbf{C}$, the pushed-forward fields are simply

$$\left(\Omega_C^{1,\bullet}, \overline{\partial}; \pi_{\mathrm{ch}} = \partial_C \colon \Omega_C^{0,\bullet} \longrightarrow \Omega_C^{1,\bullet}\right). \tag{6.27}$$

This is the Poisson BV description of the ordinary chiral boson theory on C.

6.2.4. The half-Ω theory on $C\times\Sigma$. For the fiberwise Euler field on $S = \mathrm{T}^*\Sigma$, let $i_\Sigma \colon C\times\Sigma \hookrightarrow C \times \mathrm{T}^*\Sigma$ denote the zero section. In local coordinates (z, p) the Koszul differential is contraction with $p\frac{\partial}{\partial p}$; explicitly,

$$\iota_v(\mathrm{d}z \wedge \mathrm{d}p) = -p\,\mathrm{d}z, \qquad \iota_v(a\,\mathrm{d}z + b\,\mathrm{d}p) = bp.$$

Its cohomology is therefore $\mathcal{O}_\Sigma$ in degree zero and Ω_Σ^1 in degree -1. Hence

$$\mathcal{V}^v \simeq i_{\Sigma,*}\left(\Omega_C^1 \boxtimes \left(\mathcal{O}_\Sigma \oplus \Omega_\Sigma^1[1]\right)\right) \tag{6.28}$$

as complexes of sheaves on $C \times \mathrm{T}^*\Sigma$. The right-hand side inherits the shifted Poisson structure from $\mathcal{K}_{\Omega^1}$, so this also models the localized free BV theory.

Let $p \colon \mathrm{T}^*\Sigma \longrightarrow \Sigma$ be the bundle projection and set $q = \mathbb{1}_C \times p$. Pushing the theory forward along $q \colon C \times \mathrm{T}^*\Sigma \longrightarrow C \times \Sigma$ gives the fields

$$q_*\mathscr{V}^v \simeq q_*\mathscr{K}_{\Omega^1} \simeq \Omega^{0,\bullet}\left(C \times \Sigma, \Omega_C^1 \boxtimes \left(\mathcal{O}_\Sigma \oplus \Omega_\Sigma^1[1]\right)\right). \tag{6.29}$$

Thus a field is a pair (a, b) with

$$f \in \Omega^{0,\bullet}(C \times \Sigma, \Omega_C^1 \boxtimes \mathcal{O}_\Sigma), \qquad \mathscr{b} \in \Omega^{2,\bullet}(C \times \Sigma)[1]$$

and differential $\overline{\partial}_{C\times\Sigma}$. Only the cross-pairing between the two summands is nonzero. Using Serre duality on Σ, its Poisson operator is, up to the Koszul sign from the shift,

$$\pi_{\frac{1}{2}\Omega} = \begin{pmatrix} 0 & \partial_C \\ \partial_C & 0 \end{pmatrix}. \tag{6.30}$$

$$
\begin{tikzcd}
\underline{-2} & \underline{-1} & \underline{0} & \underline{1} & \underline{2} \\
\mathcal{T}_C \boxtimes \Omega^2_S \arrow[r, "\iota_v"] & \mathcal{T}_C \boxtimes \Omega^1_S \arrow[r, "\iota_v"] & \mathcal{T}_C \boxtimes \mathcal{O}_S & & \\
 & & \mathcal{O} \arrow[r, "-(\cdot)v"] & \mathcal{O}_C \boxtimes \mathcal{T}_S \arrow[r, "-\partial_C(\cdot\wedge v)"] & \Omega^1_C \boxtimes \wedge^2\mathcal{T}_S \\
 & \mathcal{O} \arrow[r, "\cdot v"] \arrow[ur, "v\cdot"] & \mathcal{O}_C \boxtimes \mathcal{T}_S \arrow[r, "\partial_C(\cdot\wedge v)"] \arrow[ur, "{[v,-]}"] & \Omega^1_C \boxtimes \wedge^2\mathcal{T}_S \arrow[ur, "L_v"] &
\end{tikzcd}
$$

Figure 2: The holomorphic complex underlying $\widetilde{\mathcal{E}}(3|6)^v$.

It is therefore a cotangent-type family of chiral bosons over Σ; the Poisson structure differentiates only in the C-direction and is degenerate. Thus localization occurs sheafwise on the derived zero locus of v in both examples.

As a non-local action functional this theory is

$$\int_{C\times\Sigma} \mathscr{b}\,\bar\partial\partial_C^{-1} f. \tag{6.31}$$

This theory on $C \times \Sigma$ is essentially the same as the holomorphic twist of the free four-dimensional $\mathcal{N} = 1$ chiral multiplet theory [ESW20], which is presented as a higher-dimensional $\beta\gamma$-system. Indeed, if we replace f by a potential (ambiguous up to constants) $\mathscr{g} \in \Omega^{0,\bullet}(C \times S)$ satisfying $\partial_C \mathscr{g} = f$ then the action is the familiar complex two-dimensional $\beta\gamma$-system:

$$\int_{C\times S} \mathscr{b}\,\bar\partial \mathscr{g}. \tag{6.32}$$

6.3 Localization of the exceptional symmetry

Let v be a holomorphic vector field on S, regarded as the Maurer–Cartan element of Paragraph 6.1.3. The v-deformation of the exceptional symmetry is the holomorphic dg Lie algebra $\widetilde{\mathcal{E}}(3|6)^v$ on $C \times S$ with differential $[v, -]$ and the unchanged bracket of $\widetilde{\mathcal{E}}(3|6)$. Its Dolbeault resolution is the local dg Lie algebra $\widetilde{\mathscr{E}}(3|6)^v$. The underlying holomorphic complex is displayed in Figure 2.

6.3.1. The top row of Figure 2 is the dg Lie ideal

$$
(\mathcal{K}^\bullet_T, \mathrm{d} = 1 \boxtimes \iota_v) \overset{\mathrm{def}}{=} \quad
\begin{array}{ccccc}
\underline{-2} & & \underline{-1} & & \underline{0} \\
\mathcal{T}_C \boxtimes \Omega^2_S & \xrightarrow{\ \iota_v\ } & \mathcal{T}_C \boxtimes \Omega^1_S & \xrightarrow{\ \iota_v\ } & \mathcal{T}_C \boxtimes \mathcal{O}_S .
\end{array}
\tag{6.33}
$$

Its bracket combines the Jacobi bracket on $\mathcal{T}_C$ with wedge product of the negatively graded forms on S. In the notation of (6.24), the complex is a resolution:

$$\mathcal{K}^\bullet_T \simeq j_{v,*}\left(\mathcal{T}_C \boxtimes \mathcal{O}_{Z_v}\right), \tag{6.34}$$

where $\mathbf{Z}_v$ is the zero locus of v.

For $S = \mathbf{C}^2$ and $v = \mathsf{Eu}$, this gives

$$\pi_* \mathcal{K}_T^\bullet \simeq \mathcal{T}_C, \qquad \pi\colon C \times \mathbf{C}^2 \longrightarrow C. \tag{6.35}$$

For $S = \mathrm{T}^*\Sigma$ with fiberwise Euler field, set

$$\mathcal{B}_\Sigma \overset{\text{def}}{=} \mathcal{O}_\Sigma \oplus \Omega^1_\Sigma[1]. \tag{6.36}$$

This is a square-zero dg commutative algebra extension, and for $q = \mathbb{1}_C \times p\colon C \times \mathrm{T}^*\Sigma \to C \times \Sigma$ one has

$$q_* \mathcal{K}_T^\bullet \simeq \mathcal{T}_C \boxtimes \mathcal{B}_\Sigma \tag{6.37}$$

as sheaves of dg Lie algebras. These are the symmetry analogues of the two Koszul calculations in Section 6.2.

6.3.2. Let $\mathcal{Q}^v$ be the quotient by the Koszul ideal. Thus there is a short exact sequence of sheaves of dg Lie algebras

$$0 \longrightarrow \mathcal{K}_T^\bullet \longrightarrow \widetilde{\mathcal{E}}(3|6)^v \longrightarrow \mathcal{Q}^v \longrightarrow 0. \tag{6.38}$$

We record the quotient in the two cases of interest.

There is a quasi-isomorphism of sheaves of dg Lie algebras

$$\Xi\colon \Omega_C^\bullet \otimes \mathfrak{sl}(2) \longrightarrow \pi_* \mathcal{Q}^{\mathsf{Eu}}. \tag{6.39}$$

For $A \in \mathcal{O}_C \otimes \mathfrak{sl}(2)$, let

$$X_A = \sum_{i,j} A_i^j w_i \frac{\partial}{\partial w_j}.$$

The two components of Ξ are represented by

$$\Xi^0(A) = \begin{bmatrix} 0 \\ X_A \end{bmatrix}, \qquad \Xi^1(B) = \begin{bmatrix} 0 \\ X_B \wedge \mathsf{Eu} \end{bmatrix}, \quad B \in \Omega^1_C \otimes \mathfrak{sl}(2). \tag{6.40}$$

The differential identity follows from $[\mathsf{Eu}, X_A] = 0$, and the bracket identity follows from $[X_A, X_B] = X_{[A,B]}$. To see that Ξ is a quasi-isomorphism, decompose the quotient by Euler weight. Every nonzero weight is contractible; in weight zero the scalar and Euler directions form contractible pairs, and the remaining trace-free linear vector fields are precisely $\mathfrak{sl}(2)$ and their Ω^1_C-partners.

For the $\frac{1}{2}$-Ω background, define

$$\mathfrak{q}_{C,\Sigma} \overset{\text{def}}{=} \left(\Omega_C^\bullet \boxtimes \mathcal{T}_\Sigma\right) \otimes \Lambda(\epsilon), \qquad \mathrm{d} = \partial_C \otimes \mathbb{1}. \tag{6.41}$$

This naturally acts on $\mathcal{B}_\Sigma$: a vector field X acts by L_X and ϵX acts by ι_X.

There is a quasi-isomorphism of sheaves of dg Lie algebras on $C \times \Sigma$:

$$\Lambda_v\colon q_* \mathcal{Q}^v \longrightarrow \mathfrak{q}_{C,\Sigma}. \tag{6.42}$$

Indeed, let $(\cdot)^{(0)}$ denote fiber weight zero. One has $(p_*\mathcal{T}_{\mathrm{T}^*\Sigma})^{(0)} = \mathcal{A}\mathrm{t}(\mathcal{T}_\Sigma)$; write $\sigma\colon \mathcal{A}\mathrm{t}(\mathcal{T}_\Sigma) \to \mathcal{T}_\Sigma$ for its symbol. Wedging with v gives an isomorphism

$$v \wedge -\colon \mathcal{T}_\Sigma \xrightarrow{\cong} \left(p_* \wedge^2 \mathcal{T}_{\mathrm{T}^*\Sigma}\right)^{(0)},$$

and let κ denote its inverse. On the nonzero components of the quotient, the map is

$$\begin{aligned} \Lambda_v^0 \begin{bmatrix} X \\ h \end{bmatrix} &= \sigma(X^{(0)}), \\ \Lambda_v^1 \begin{bmatrix} Y \\ \rho \end{bmatrix} &= \kappa(\rho^{(0)}) + \epsilon\sigma(Y^{(0)}), \\ \Lambda_v^2(\nu) &= \epsilon\kappa(\nu^{(0)}). \end{aligned} \tag{6.43}$$

All nonzero fiber weights are contractible. In weight zero, the scalar and vertical Euler directions again cancel, leaving exactly (6.41). This proves that Λ_v is a quasi-isomorphism. The symbol map, κ, and the Cartan identities also identify the induced quotient bracket with the bracket on $\mathfrak{q}_{C,\Sigma}$.

6.3.3. Ω-background localization on C. Combining these observations in the case of $S = \mathbf{C}^2$, $v = \mathsf{Eu}$ gives the full localized symmetry on the curve.

Proposition 6.7. *Let $\pi\colon C\times\mathbf{C}^2 \to C$ and $v = \mathsf{Eu}$. There are quasi-isomorphisms of sheaves of dg Lie algebras on C*

$$\pi_*\widetilde{\mathcal{E}}(3|6)^{\mathsf{Eu}} \simeq \mathcal{T}_C \ltimes \left(\Omega_C^\bullet \otimes \mathfrak{sl}(2)\right) \simeq \mathcal{T}_C \oplus \underline{\mathfrak{sl}(2)}_C. \tag{6.44}$$

Here $\mathcal{T}_C$ acts on the holomorphic de Rham complex by Lie derivative, and $\underline{\mathfrak{sl}(2)}_C$ denotes the locally constant sheaf.

Proof. The map to $\pi_*\widetilde{\mathcal{E}}(3|6)^{\mathsf{Eu}}$ is the constant-in-S inclusion on $\mathcal{T}_C$ and is represented by (6.40) on the $\mathfrak{sl}(2)$ summand. It is a strict dg Lie map. The vertical linear vector fields realize the $\mathfrak{sl}(2)$ bracket, and the mixed bracket is the Lie derivative. Comparing the short exact sequence (6.38) with

$$0 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{T}_C \ltimes (\Omega_C^\bullet \otimes \mathfrak{sl}(2)) \longrightarrow \Omega_C^\bullet \otimes \mathfrak{sl}(2) \longrightarrow 0,$$

the maps on the ideal and quotient are the quasi-isomorphisms (6.35) and (6.39). Hence the middle map is a quasi-isomorphism. Finally, the holomorphic Poincaré lemma identifies $\Omega_C^\bullet$ with the constant sheaf $\underline{\mathbf{C}}_C$; constant $\mathfrak{sl}(2)$-valued sections have trivial $\mathcal{T}_C$ action, giving the second quasi-isomorphism in (6.44). ◻

6.3.4. The $\frac{1}{2}$-Ω localization on $C \times \Sigma$. Let $S = \mathrm{T}^*\Sigma$, let v be the fiberwise Euler field, and let $q = \mathbb{1}_C \times p$. The fiber weight decomposition is compatible with the differential and bracket on $q_*\widetilde{\mathcal{E}}(3|6)^v$.

Proposition 6.8. *The inclusion of the fiber-weight-zero subalgebra is a quasi-isomorphism of sheaves of dg Lie algebras*

$$\left(q_*\widetilde{\mathcal{E}}(3|6)^v\right)^{(0)} \xrightarrow{\simeq} q_*\widetilde{\mathcal{E}}(3|6)^v. \tag{6.45}$$

There is an exact sequence of sheaves of dg Lie algebras on $C \times \Sigma$:

$$0 \to \mathcal{T}_C \boxtimes \mathcal{B}_\Sigma \longrightarrow q_* \widetilde{\mathcal{E}}(3|6)^v \longrightarrow \mathfrak{q}_{C,\Sigma} \to 0, \tag{6.46}$$

where $\mathcal{B}_\Sigma$ and $\mathfrak{q}_{C,\Sigma}$ are defined in (6.36) and (6.41).

Proof. Filter the complex by the three rows in Figure 2 and then decompose by fiber weight. On the Koszul row, the Cartan identity $[\partial_S, \iota_v] = L_v$ makes $k^{-1}\partial_S$ a contracting homotopy in every nonzero weight k. On the associated graded of the quotient rows, $[v, -]$ and L_v act by the same nonzero weight; the remaining arrows preserve this filtration. The resulting spectral sequence therefore has only its weight-zero part, proving (6.45). Applying q_* to (6.38), the ideal is identified with $\mathcal{T}_C \boxtimes A_\Sigma$ by (6.37), while the quotient is identified with $\mathfrak{q}_{C,\Sigma}$ by (6.42). This gives (6.46). □

The first term of (6.46) consists of C-vector fields with coefficients in the derived zero locus:

$$\mathcal{T}_C \boxtimes \mathcal{O}_\Sigma \oplus \mathcal{T}_C \boxtimes \Omega^1_\Sigma[1].$$

The quotient consists of $\Omega^\bullet_C$-valued Lie derivatives and contractions on A_Σ. At the level of the reduced underlying complex, the unshifted degree-zero terms combine as

$$\mathcal{T}_C \boxtimes \mathcal{O}_\Sigma \oplus \mathcal{O}_C \boxtimes \mathcal{T}_\Sigma = \mathcal{T}_{C\times\Sigma};$$

the remaining terms provide its de Rham and derived partners. This is the exceptional symmetry of the degenerate Poisson BV theory on $C \times \Sigma$ described in Paragraph 6.2.4. We will return to the corresponding pushforward of factorization algebras below.

6.4 Dolbeault conformal structures on factorization algebras

A conformal structure on a vertex algebra arises from the stress tensor in the conformal field theory. One way to phrase a conformal structure is the data of a map of vertex algebras from the Virasoro vertex algebra, at some value of the central charge, into the vertex algebra. This definition also makes perfect sense at the level of factorization algebras.

Consider the local dg Lie algebra defined on a Riemann surface C

$$\mathscr{T}_C \stackrel{\text{def}}{=} \Omega^{0,\bullet}(C, \mathcal{T}_C) \tag{6.47}$$

which is equipped with the $\overline{\partial}$ differential and the Jacobi bracket of Dolbeault-valued sections of the holomorphic tangent bundle. It is shown in [Wil17a; Wil24c] that $H^1_{loc}(\mathscr{T}_C) \simeq \mathbf{C}$ which is locally represented by the cocycle

$$\theta_{Vir} \colon (f\partial_z, g\partial_z) \mapsto \frac{1}{12}\frac{1}{2\pi i}\int_{\mathbf{C}_z} \partial_z f \partial \partial_z g \tag{6.48}$$

This is, of course, a local form of the residue formula for the cocycle defining the Virasoro algebra. Indeed, when $C = \mathbf{C}$, the twisted enveloping factorization algebra

$$\mathbf{U}_{c\theta_{Vir}}(\mathscr{T}_{\mathbf{C}}) \tag{6.49}$$

corresponds to the Virasoro vertex algebra of central charge $c \in \mathbf{C}$ [Wil17a].

Let $\mathcal{F}$ be a factorization algebra on a Riemann surface C. We define a *conformal structure* on $\mathcal{F}$ to be a map of factorization algebras

$$\mathsf{T}\colon \mathbf{U}_{c\theta_{Vir}}(\mathcal{T}_C) \to \mathcal{F}. \tag{6.50}$$

In the case $C = \mathbf{C}$, and when $\mathcal{F}$ satisfies the conditions of [CG17, section 5.3.2], this data induces a conformal structure on the vertex algebra $\mathbb{V}(\mathcal{F})$ of central charge c.

The exceptional Lie superalgebra $E(3|6)$ (respectively $\widetilde{E}(3|6)$) underlies the symmetries of the stress tensor in the minimal holomorphic twist (respectively, alternative holomorphic twist) of a six-dimensional $\mathcal{N} = (2,0)$ superconformal field theory. This motivates the following definition.

Definition 6.9. Let C be a complex curve and S a complex surface. A *Dolbeault-conformal* structure on a factorization algebra $\mathcal{F}$ on $C \times S$ is a map of factorization algebras

$$\mathsf{T}\colon \mathbf{U}_{c_{fr}\Theta_{fr}}(\widetilde{\mathcal{E}}(3|6)) \to \mathcal{F} \tag{6.51}$$

where $\widetilde{\Theta}_{fr} \in C^\bullet_{loc}(\widetilde{\mathcal{E}}(3|6))$ is the local cocycle of total degree $+1$ from proposition 5.6. We refer to $c_{fr} \in \mathbf{C}$ as the free central charge of $\mathcal{F}$.

Following the work of [Nak97] and [Gro96], as well as motivations from string theory, we know that the sum of de Rham cohomologies of the Hilbert schemes of points is a conformal vertex algebra. We use our current terminology to refer to the expectation that this structure is present in the *Dolbeault* cohomology (rather than the de Rham cohomology) of the Hilbert scheme of points on S, since we are studying the holomorphic twist rather than the more topological one relevant in the works *loc. cit.*.

6.4.1. We have two main examples of Dolbeault-conformal factorization algebras:

- The factorization algebra $\mathrm{Obs}_{\mathcal{R}}$ underlying the observables of the holomorphic twist of the free $\mathcal{N} = (2,0)$ theory is an $E(3|6)$-factorization algebra of six-dimensional central charge 1. The factorization algebra $\widetilde{\mathrm{Obs}}$ is a Dolbeault Virasoro factorization algebra of six-dimensional central charge 1.
- Tautologically, the factorization algebra $\mathbf{U}_{c\Theta_{fr}}(\mathcal{E}_{(3|6)})$ is an $E(3|6)$ factorization algebra for any $c \in \mathbf{C}$. Likewise, the factorization algebra $\mathbf{U}_{c\widetilde{\Theta}_{fr}}(\widetilde{\mathcal{E}}_{(3|6)})$ is a Dolbeault Virasoro factorization algebra for any $c \in \mathbf{C}$.

6.5 The main construction: localizing to a curve

Let C be a complex curve, let S be a complex surface, and write $\pi\colon C \times S \to C$ for the projection. Fix a holomorphic vector field v on S whose zero locus consists of a single point. Let $(\mathcal{F}, \mathbf{T})$ be a Dolbeault-conformal factorization algebra on $C \times S$; thus $\mathcal{F} = (\mathcal{F}, \mathrm{d}_{\mathcal{F}})$ is a factorization algebra valued in cochain complexes and $\mathbf{T}$ is its Dolbeault stress tensor.

Evaluating the current on the Maurer–Cartan element $v \in \Gamma(\mathcal{O}_C \boxtimes \mathcal{T}_S)$ gives a deformation

$$\mathcal{F}^v \overset{\mathrm{def}}{=} (\mathcal{F}, \mathrm{d}_{\mathcal{F}} + \mathbf{T}(v) \star (-))\,. \tag{6.52}$$

The equivariant master equation for $\mathbf{T}$ implies that the differential in (6.52) squares to zero. Since the current action is compatible with factorization products, $\mathcal{F}^v$ is again a factorization algebra on $C \times S$.

Definition 6.10. The localization of $(\mathcal{F}, \mathbf{T})$ along the pair (π, v) is the factorization algebra on C

$$\mathrm{Loc}_{\pi,v}(\mathcal{F}, \mathbf{T}) \stackrel{\text{def}}{=} \pi_* \mathcal{F}^v. \tag{6.53}$$

Equivalently, for every open set $U \subset C$,

$$\mathrm{Loc}_{\pi,v}(\mathcal{F}, \mathbf{T})(U) = \left(\mathcal{F}(U \times S), \mathrm{d}_{\mathcal{F}} + \mathbf{T}(v) \star (-)\right). \tag{6.54}$$

For pairwise disjoint opens $U_1, \ldots, U_n \subset U$, its factorization product is the original structure map

$$\bigotimes_{i=1}^{n} \mathcal{F}(U_i \times S) \longrightarrow \mathcal{F}(U \times S). \tag{6.55}$$

A morphism of Dolbeault-conformal factorization algebras intertwines the currents, hence gives a cochain map in (6.54); this specifies the functor on morphisms. When the current is understood as part of the structure, we write $\mathrm{Loc}_{\pi,v}(\mathcal{F})$.

We will use this construction only when $S = \mathbf{C}^2$ and

$$v = \mathsf{Eu} \stackrel{\text{def}}{=} w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2}, \tag{6.56}$$

whose zero locus is $\{0\}$. In this case (6.53) specializes to the functor

$$\mathrm{Loc}_{\pi,\mathsf{Eu}} \colon \mathrm{Fact}^{(3|6)}(C \times \mathbf{C}^2) \longrightarrow \mathrm{Fact}^{\mathrm{conf}}(C) \tag{6.57}$$

announced in (6.1).

We prove the following

Theorem 6.11. *Let $\mathcal{F}$ be a Dolbeault-conformal factorization algebra on $C \times \mathbf{C}^2$ of free central charge c_{fr}. Then $\mathrm{Loc}_{\pi,\mathsf{Eu}}(\mathcal{F})$ has the structure of a chiral conformal factorization algebra on C of central charge $c_{2d} = c_{fr}$.*

Remark 6.12. In upcoming work with Gui, we have shown that locally on $\mathbf{C}\times\mathbf{C}^2$ the space of anomaly classes for $E(3|6)$ is, in fact, two-dimensional:

$$H^1_{loc}(\mathcal{E}(3|6)_{\mathcal{R}}) \cong H^1_{loc}(\widetilde{\mathcal{E}}(3|6)) \cong H^7(E(3|6)) \cong \mathbf{C}^2.$$

One direction is spanned $\theta_{fr} = [\Theta_{fr}]$ and hence detected by c_{fr}. Let's choose an inedendent basis element θ_{Lou}. We postpone an analysis about how the other class localizes to correct the two-dimensional central charge c_{2d}. In the case of the more general Ω-background deformation $v_{\varepsilon_1,\varepsilon_2}$ of remark 6.2, the localized free central charge is independent of the ratio $\frac{\varepsilon_1}{\varepsilon_2}$ whereas the result of localizing θ_{Lou} depends on it.

Let C be a Riemann surface. Recall that $\mathbf{U}_{\widetilde{\Theta}_{fr}}(\widetilde{\mathcal{E}}(3|6))$ is enveloping factorization algebra twisted by the local cocycle $\widetilde{\Theta}_{fr}$. By definition, this is a Dolbeault-conformal factorization algebra on $C \times \mathbf{C}^2$. The proof of this theorem amounts to showing that the pushforward along π of this factorization algebra is, itself, a conformal factorization algebra.

6.5.1. The v-deformation of the enveloping factorization algebra $\mathbf{U}(\widetilde{\mathcal{E}}(3|6))$ is equal to the enveloping factorization algebra of the v-deformed local Lie algebra $\widetilde{\mathcal{E}}(3|6)^v$ as described in figure 2.

Before including the central extension, we conclude from 6.3.1 that there is an embedding of enveloping factorization algebras

$$\mathbf{U}\left(\mathcal{K}_T\right) \hookrightarrow \mathbf{U}\left(\widetilde{\mathcal{E}}(3|6)\right)^v \tag{6.58}$$

The right hand side denotes the v-deformation of the factorization algebra $\mathbf{U}\left(\widetilde{\mathcal{E}}(3|6)\right)$. On the left-hand side appears $\mathcal{K}_T$, the Dolbeault complex of the holomorphic dg Lie algebra $\mathcal{K}_T$ introduced in 6.33. Explicitly

$$\mathcal{K}_T = \Omega^{-\bullet,\bullet}_{\mathbf{C}^2} \boxtimes \mathcal{T}_C$$

with the differential $\overline{\partial}_{\mathbf{C}^2} + \iota_v + \overline{\partial}_C$. The bracket uses the Jacobi bracket of vector fields on C together with the wedge product of (negatively graded) forms on $\mathbf{C}^2$. Thus, if $\pi \colon \mathbf{C}^2 \times C \to C$ is projection, then we have an equivalence of sheaves on C:

$$\pi_*(\mathcal{K}_T) \simeq \mathcal{T}^1_C. \tag{6.59}$$

Thus, in the case of vanishing central charge, we see that

$$\pi_* \mathbf{U}(\mathcal{K}_T) \simeq \mathbf{U}\mathcal{T}_C. \tag{6.60}$$

6.5.2.

It remains to prove the compatibility of the local cocycle of $\mathcal{T}_C$ which defines the familiar Virasoro central extension with the behavior of the $\widetilde{\mathcal{E}}(3|6)$ anomaly $\widetilde{\Theta}_{fr}$ after the v-deformation. This calculation is the main content of the remainder of this section, which we summarize here.

Proposition 6.13. *There is an equivalence of factorization algebras*

$$\mathrm{Loc}_{\pi,\mathrm{Eu}}\left(\mathbf{U}_{\widetilde{\Theta}_{fr}}(\widetilde{\mathcal{E}}(3|6))\right) \simeq \mathbf{U}_{\theta_{Vir}}(\mathcal{T}_C) \otimes \mathbf{U}(\mathfrak{sl}(2) \otimes \Omega^\bullet_C). \tag{6.61}$$

In other words, the localization of the factorization algebra $\mathbf{U}_{c\widetilde{\Theta}_{fr}}(\widetilde{\mathcal{E}}(3|6))$ is the Virasoro factorization algebra of central charge c tensored with the factorization enveloping algebra of the locally constant sheaf of Lie algebras $\mathfrak{sl}(2) \otimes \Omega^\bullet_C$ equipped with the full (complexified) de Rham differential $\mathrm{d}_{dR} = \overline{\partial}_C + \partial_C$.

This proposition immediately implies theorem 6.11.

Remark 6.14. Locally, on $C = \mathbf{C}$, we will recall how to extract vertex algebra from the data of a holomorphic factorization algebra in section 6.8. In the case at hand, $\mathbf{U}_{c\theta_{Vir}}(\mathcal{T}_{\mathbf{C}})$ produces the Virasoro vertex algebra of central charge c. The topological piece $\mathbf{U}(\mathfrak{sl}(2) \otimes \Omega^\bullet_{\mathbf{C}})$ produces a commutative vertex algebra whose underlying vector space is $\wedge^\bullet(\mathfrak{sl}(2))$. In fact, on $\mathbf{C} = \mathbf{R}^2$ the locally constant factorization algebra $\mathbf{U}(\mathfrak{sl}(2) \otimes \Omega^\bullet_{\mathbf{C}})$ is a nontrivial $\mathbb{E}_2$-algebra called the $\mathbb{E}_2$-enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$, see [Knu18]. Thus, on $C = \mathbf{C}$ this pushforward produces the tensor product of the Virasoro vertex algebra with the $\mathbb{E}_2$-enveloping algebra of $\mathfrak{sl}(2)$.

6.6 Anomaly and the central charge

Recall that $\widetilde{\Theta}_{fr}$ is defined to be the central anomaly for the quantization of the classical $\widetilde{\mathcal{E}}_{(3|6)}$ background $\widetilde{\mathbf{T}}_{(3|6)}$ on the polarized holomorphic twist of the $\mathcal{N} = (2,0)$ theory. To relate $\widetilde{\Theta}_{fr}$ to the usual two-dimensional central charge, defined in terms of local cocycle θ_{Vir}, we follow the following steps.

(a) Let $\mathcal{K}_T$ be the Dolbeault complex of the holomorphic dg Lie algebra $\mathcal{K}_T$ defined in (6.33) and denote the natural inclusion

$$j \colon \mathcal{K}_T \to \widetilde{\mathcal{E}}(3|6)^v \tag{6.62}$$

Describe $j^*\widetilde{\Theta}_{fr}$.

(b) We have already argued in 6.7 that $\pi_*\widetilde{\mathcal{E}}(3|6)^v$ contains the local dg Lie algebra $\mathcal{T}_C$. In fact, we have shown that there is a quasi-isomorphism of sheaves of cochain complexes $\mathcal{T}_C \simeq \pi_*\mathcal{K}_T$. We study how this intertwines the natural Lie brackets. Namely, we produce an explicit L_∞-quasi-isomorphism of precosheaves of dg Lie algebras

$$\Psi \colon \mathcal{T}_{C,c} \xrightarrow{\simeq} (\pi_*\mathcal{K}_T)_c \tag{6.63}$$

In other words, for each open set $U \subset C$ we have an L_∞-quasi-isomorphism

$$\Psi(U) \colon \Omega^{0,\bullet}_c(U, \mathrm{T}) \xrightarrow{\simeq} \Omega^{-\bullet,\bullet}(\mathbf{C}^2) \mathbin{\widehat{\otimes}} \Omega^{0,\bullet}_c(U, \mathrm{T}) \tag{6.64}$$

where the differential on the right hand side is $\bar{\partial} + \iota_v$.

(c) Show that $\theta_{Vir} = \Psi^* j^*\widetilde{\Theta}_{fr}$.

Let $\mu_z = \mu_z^0 + \mu_z^1 + \mu_z^2$ be a section of $\mathcal{K}_T$ where μ_z^i is of holomorphic Dolbeault type i along $\mathbf{C}^2$.

6.6.1. The first item is accomplished by the following proposition which explicitly characterizes the anomaly when restricted to $\mathcal{K}_T$.

Proposition 6.15. *Let $U_z \subset C$ be a coordinatized open set. The local cocycle $j^*\widetilde{\Theta}_{fr}$ for the local dg Lie algebra $\mathcal{K}_T$ on $\mathbf{C}^2 \times U_z$ is cohomologous to*

$$\frac{1}{(2\pi \mathrm{i})^3}\frac{1}{12}\int J\mu_z \wedge \partial_C J\mu_z \tag{6.65}$$

Proof. The proof of this is an explicit Feynman diagram computation which we will perform on $\mathbf{C}^2 \times \mathbf{C} = \mathbf{C}^3$. By restriction, the current $\widetilde{\mathbf{T}}$ provides a $\mathcal{K}_T$-background for the polarized twist of the $\mathcal{N} = (2,0)$ theory after further deforming by the Maurer–Cartan element S. We reference a simple equivalent description of the v-deformation of the theory from 6.2. In short, the space of fields of the free theory is the Dolbeault complex of the complex of holomorphic bundles

$$\mathcal{K}_{\Omega^1} = \left(\Omega^1_C \boxtimes \Omega^{-\bullet}_{\mathbf{C}^2},\ \mathrm{d} = \iota_v\right), \tag{6.66}$$

which is concentrated in degrees $-2,-1,0$. We will write fields as $\alpha = \alpha^0 + \alpha^1 + \alpha^2$ where $\alpha^{i=0,1,2}$ denotes the degree of the holomorphic form component along $\mathbf{C}^2$. The action functional of this free theory is

$$\int_{\mathbf{C}^2 \times C} \partial_C^{-1}\alpha\left(\overline{\partial}\alpha + \iota_v \alpha\right). \tag{6.67}$$

The non-local action functional reflects the fact that the theory is Poisson degenerate. The $\mathcal{K}_T$-background of this theory takes the simple form

$$\mathbf{T}_{\mathcal{K}_T}(\mu) = \frac{1}{2}\int_{\mathbf{C}^2\times C} \alpha(\iota_\mu \alpha) = \int_{\mathbf{C}^2 \times C}\left(\alpha^0\mu^0 \vee \alpha^2 + \frac{1}{2}\alpha^1\mu^0 \vee \alpha^1 + \frac{1}{2}\alpha^0\mu^2 \vee \alpha^0 + \alpha^0\mu^1 \vee \alpha^1\right) \tag{6.68}$$

where μ denotes a section of $\mathcal{K}_T$.

Feynman diagram weights, like the ones which compute the anomaly, are defined using the propagator of the theory. Formally, when we consider the graph expansion, it is convenient to treat the term $\iota_v\alpha$ in the kinetic term of the action as a (nilpotent) vertex. Therefore, the edges of Feynman graphs are labeled by the propagator $\partial_C P_{\overline{\partial}}$ associated to the first term in the action (6.67). Here, $P_{\overline{\partial}}$ is the Green's function for the (total) $\overline{\partial}$ operator on $\mathbf{C}^2 \times C_z$.

From proposition 5.4, we know that the anomaly will be a sum of weights assigned to wheel graphs with four vertices. The only nonzero contributions are from those wheels where precisely two of the the vertices are labeled by the special vertex S (and so there is no outgoing edge at each of these vertices) and two are labeled by $\mathbf{T}_{\mathcal{K}_T}$ whose external edges are labeled by μ. We refer to figure 3.

To compute the weight we use heat kernel regularization following [Cos11; Wil20; GW18]. Denote a coordinate on $\mathbf{C}^3$ by $X = (x_1, x_2, z)$ and label the vertices $0,1,2,3$ as in the figure. If $\mu_0 = \alpha\partial_z, \mu_3 = \alpha\partial_z$ denote compactly supported fields inserted at the external edges, then the weight of this wheel diagram is explicitly computed as the $L \to 0$ and $\epsilon \to 0$ limit of the integral

$$\int_{(\mathbf{C}^3)^4} \partial_z\alpha(X^0)P_{\epsilon<L}(X^0,X^1)i_{S(x^1)}P_{\epsilon<L}(X^1-X^2)\iota_{S(x^2)}P_{\epsilon<L}(X^2-X^3)K_{\epsilon<L}(X^3-X^0)\partial_z\alpha(X^3). \tag{6.69}$$

Here, $P_{\epsilon<L}$ is the cutoff propagator for the $\overline{\partial}$-operator. Following the appendix in [GW18], we make the change of coordinates

$$\begin{aligned} X &= X^0 \\ \widetilde{X}^i &= X^i - X^{i-1}, \quad i = 1,2,3. \end{aligned}$$

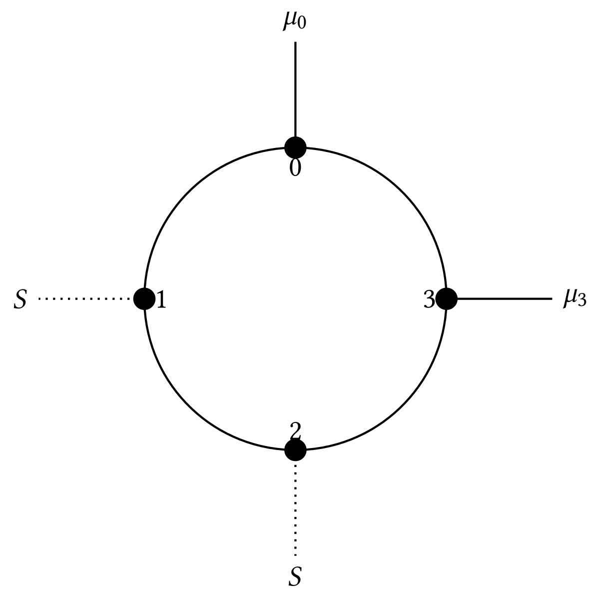

Figure 3: The anomaly wheel

Simplifying as in *loc. cit.* we find that the weight can be written as the integral

$$\int_X \int_{\widetilde{X}^1,\widetilde{X}^2,\widetilde{X}^3} \mathscr{A} \frac{1}{(2\pi \mathrm{i})^6} \int_{t_1,t_2,t_3=\epsilon}^{L} \frac{1}{t_1^4 t_2^4 t_3^4} \epsilon_{i_1 i_2 i_3} \epsilon_{ij} \widetilde{x}_i^1 \widetilde{x}_j^2 \overline{\widetilde{X}}^1 \overline{\widetilde{X}}^2 \overline{\widetilde{X}}^3 e^{-M_{ab} \widetilde{X}^i \cdot \overline{\widetilde{X}}^i}, \tag{6.70}$$

where $\mathscr{A} = \partial_{\overline{z}^0} \alpha \partial_{\overline{z}^3} \alpha$. Evaluating the $\widetilde{X}^1, \widetilde{X}^2, \widetilde{X}^3$ Gaussian integrals we find that the term which is lowest order in ϵ, L (and the only one which survives the $\epsilon, L \longrightarrow 0$ limits) is

$$2 \cdot \frac{1}{(2\pi \mathrm{i})^3} \frac{1}{4!} \int_{X \in \mathbf{C}^3} \partial_z \alpha(X) \partial \partial_z \alpha(X) + \cdots \tag{6.71}$$

The factor of 2 is from the fact that there are two marked vertices. The factor of 4! arises from the explicit Schwinger space integral over t_1, t_2, t_3. The $\cdots$ stand for terms which are proportional to w_1 or w_2. As local functionals, these terms are all cohomologically trivial.

□

6.6.2. We proceed to item (2) highlighted in the beginning of section 6.6. Introduce a smooth function $\rho\colon \mathbf{C}^2 \longrightarrow \mathbf{C}$ satisfying the properties:

(a) ρ is compactly supported and is identically 1 in a small neighborhood of 0.

(b) ρ is radially symmetric (only depends on $R^2 = |w|^2$) and vanishes on a sufficiently large sphere centered at the origin.

The essential bit is item (1).

Lemma 6.16. *Suppose that $U \subset C$ is an open set. There is an L_∞ quasi-isomorphism of dg Lie algebras*

$$\Psi_U : \Omega_c^{0,\bullet}(U, \mathrm{T}) \rightsquigarrow \mathcal{K}_{T,c}(\mathbf{C}^2 \times U) \tag{6.72}$$

whose linear component is

$$\Psi_U^{(1)}(\mu) = \left(\rho - \frac{\overline{w}_1 \mathrm{d}w_1 + \overline{w}_2 \mathrm{d}w_2}{\|w\|^2} \overline{\partial}\rho + \mathrm{d}^2 w\, \omega\, \overline{\partial}\rho\right)\mu \tag{6.73}$$

where

$$\omega \stackrel{\text{def}}{=} \frac{\overline{w}_1 \mathrm{d}\overline{w}_2 - \overline{w}_2 \mathrm{d}\overline{w}_1}{\|w\|^4} \tag{6.74}$$

and $\|w\|^2 = w_1\overline{w}_1 + w_2\overline{w}_2$.

Proof. We show that $\Psi^{(1)} = \Psi_U^{(1)}$ is a cochain map. To this end, define the differential form

$$F(\rho) \stackrel{\text{def}}{=} \left(\rho - \frac{\overline{w}_1 \mathrm{d}w_1 + \overline{w}_2 \mathrm{d}w_2}{\|w\|^2} \overline{\partial}\rho + \mathrm{d}^2 w\, \omega\, \overline{\partial}\rho\right). \tag{6.75}$$

We can view $F(\rho)$ as a compactly supported section of the $\mathbf{Z}$-graded vector bundle $\wedge^\bullet \overline{\mathrm{T}}^* \otimes \wedge^{-\bullet}\mathrm{T}^*$ on $\mathbf{C}^2$; in other words $F \in \Omega_c^{-\bullet,\bullet}(\mathbf{C}^2)$. As such an element it is of cohomological degree zero. By direct computation one can verify that $F(\rho)$ solves the following differential equation

$$\overline{\partial}F(\rho) + \iota_v F(\rho) = 0, \tag{6.76}$$

as compactly supported differential forms on $\mathbf{C}^2$.

The form F defines a quasi-isomorphism of $\mathbf{C}$ (concentrated in degree zero) and

$$\begin{array}{ccccc} \underline{-2} & & \underline{-1} & & \underline{0} \\ \Omega_c^{2,\bullet}(\mathbf{C}^2) & \xrightarrow{\iota_v} & \Omega_c^{1,\bullet}(\mathbf{C}^2) & \xrightarrow{\iota_v} & \Omega_c^{0,\bullet}(\mathbf{C}^2). \end{array} \tag{6.77}$$

In one direction, the map is simply given by the element $F(\rho)$. The other direction is simply integration.

Define, for $\mu \in \Omega_c^{0,\bullet}(U, \mathrm{T})$ the section

$$\Psi^{(1)}(\mu) = F(\rho)\mu. \tag{6.78}$$

So, in local coordinates, if $\mu = f(z)\partial_z$ then $\Psi(f(z)\partial_z) = F(\rho)f(z)\partial_z$, which is now thought of as a compactly supported Dolbeault valued holomorphic vector field on $\mathbf{C}^2 \times U$.

Moreover, for $\mu \in \Omega_c^{0,\bullet}(U, \mathrm{T})$ we have, by definition

$$\Psi^{(1)}(\overline{\partial}\mu) = F(\rho)\overline{\partial}\mu. \tag{6.79}$$

On the other hand since $F(\rho)$ is an even differential form we have

$$(\overline{\partial} + \iota_v)\Psi^{(1)}(\mu) = (\overline{\partial}F(\rho) + \iota_v F(\rho))\mu + F(\rho)\overline{\partial}\mu = F(\rho)\overline{\partial}\mu, \tag{6.80}$$

thus verifying that $\Psi^{(1)}$ is a cochain map.

Notice that the failure for $\Psi^{(1)}$ to be a (strict) Lie algebra homomorphism is

$$[\Psi^{(1)}(\mu), \Psi^{(1)}(\mu')] = F(\rho)(F(\rho) - 1)[\mu, \mu']. \tag{6.81}$$

To define the homotopy we use the differential form

$$G(\rho) \overset{\text{def}}{=} \rho(\rho - 1)\left(\frac{\overline{w}_1 \mathrm{d}w_1 + \overline{w}_2 \mathrm{d}w_2}{\|w\|^2} + \omega \mathrm{d}^2 w\right). \tag{6.82}$$

One can check that $G(\rho)$ solves the differential equation

$$(\overline{\partial} + S \vee -)G(\rho) = F(\rho)(F(\rho) - 1). \tag{6.83}$$

In particular $G(\rho)$ is of cohomological degree -1 in the complex $\Omega_c^{-\bullet,\bullet}(\mathbf{C}^2)$. Thus, we see that the 2-ary morphism should be given by

$$\Psi^{(2)}(\mu, \mu') = G(\rho)[\mu, \mu']. \tag{6.84}$$

One can check the remaining L_∞ relations are satisfied. □

6.6.3. Finally, we compute $\Psi^* j^* \widetilde{\Theta}_{fr}$ Suppose that $U_z \subset C$ is a coordinatized open subset and let f, g be compactly supported $(0, \bullet)$-forms suppored on U_z. The form f defines the compactly supported Dolbeault-valued vector field $f(z, \overline{z})\partial_z$ on U_z. Then, we use the quasi-isomorphism $\Psi^{(1)}$ to obtain a compactly supported field $\Psi^{(1)}(f(z, \overline{z})\partial_z)$ on $\mathbf{C}^2 \times U_z$. We then see the value of the local cocycle $\Psi^* j^* \widetilde{\Theta}_{fr}$ evaluated on the pair $\Psi^{(1)}(f\partial_z), \Psi^{(1)}(g\partial_z)$ is

$$\frac{1}{(2\pi \mathrm{i})^3} \frac{1}{12} \int_{\mathbf{C}^2 \times U_z} \mathrm{d}^2 w \omega \rho \overline{\partial} \rho \partial_z f(z) \partial \partial_z g(z) = \frac{1}{2\pi \mathrm{i}} \frac{1}{12} \int_{U_z} \partial_z f \partial \partial_z g. \tag{6.85}$$

We have used the following computation

$$\begin{aligned}
\int_{\mathbf{C}^2} \rho \overline{\partial} \rho \mathrm{d}^2 w \omega &= \frac{1}{2} \int_{\mathbf{C}^2} \overline{\partial}(\rho^2) \mathrm{d}^2 w \omega \\
&= \lim_{R \to \infty} \frac{1}{2} \int_{B_R} \overline{\partial}(\rho^2) \mathrm{d}^2 w \omega \\
&= (2\pi \mathrm{i})^2 - \lim_{R \to \infty} (2\pi \mathrm{i})^2 \oint_{S^3_R} \rho^2 \mathrm{d}^2 w \omega \\
&= (2\pi \mathrm{i})^2.
\end{aligned}$$

Here B_R denotes a a four-ball of radius R centered at the origin and $S^3_R = \partial B_R$. In the last line we have used the Bochner–Martinelli residue formula which states that for any compactly supported smooth function $f : \mathbf{C}^2 \to \mathbf{C}$ and any $R > 0$ that

$$f(0) = \int_{D_R} \overline{\partial} f \omega_{BM} - \oint_{S^3_R} f \omega_{BM} \tag{6.86}$$

where $\omega_{BM} = \frac{2}{(2\pi \mathrm{i})^2} \mathrm{d}^2 w \omega$, and ω is the $(0, 1)$-form in equation (6.74). Also note that the integral over the sphere vanishes as $R \to \infty$ by our hypothesis on the function ρ. This completes the proof of proposition 6.13.

6.7 The localized free factorization algebra

We now apply the preceding localization to the factorization algebra of quantum observables of the polarized free theory. Let Bos denote the free BV quantization of the chiral boson theory on C, whose fields and (-1)-shifted Poisson structure are

$$\Omega^{1,\bullet}_C, \qquad \mathrm{d} = \overline{\partial}, \qquad \pi_{\mathrm{ch}} = \partial_C \colon \Omega^{0,\bullet}_C \longrightarrow \Omega^{1,\bullet}_C. \tag{6.87}$$

Thus, for $U \subset C$,

$$\mathrm{Bos}(U) = \left(\mathrm{Sym}\big(\Omega^{0,\bullet}_c(U)[1]\big), \overline{\partial} + \triangle_{\mathrm{ch}}\right), \tag{6.88}$$

where the BV Laplacian is characterized on quadratic observables by

$$\triangle_{\mathrm{ch}}(b_1 b_2) = \int_U b_1 \partial_C b_2. \tag{6.89}$$

This is an instance of the Heisenberg factorization algebra construction of [CG17, section 2.3].

Theorem 6.17. *Let $v = \mathsf{Eu}$ on $\mathbf{C}^2$ and let $\pi\colon C \times \mathbf{C}^2 \to C$. The localization of the factorization algebra $\widetilde{\mathrm{Obs}}$ of the polarized free theory is equivalent to the chiral boson factorization algebra:*

$$\mathrm{Loc}_{\pi,\mathsf{Eu}}(\widetilde{\mathrm{Obs}}) = \pi_*\left(\widetilde{\mathrm{Obs}}^{v}\right) \simeq \mathrm{Bos}\,. \tag{6.90}$$

This is an equivalence of chiral conformal factorization algebras, with central charge one.

Proof. The v-background is quadratic, so $\widetilde{\mathrm{Obs}}^{v}$ is the free BV quantization of the deformed theory $\mathscr{V}^{v}$. Proposition 6.6 and (6.26) identify the pushforward of this Poisson BV theory on $C \times \mathbf{C}^2$ with the chiral boson theory (6.87) supported on C.

For completeness, filter both factorization algebras by polynomial degree in the observables. On the associated graded, the equivalence is the symmetric algebra on the density-valued dual of the field quasi-isomorphism. Equation (6.21) says precisely that this map intertwines the two quadratic Poisson cocycles, and hence the two BV Laplacians. After normalizing the Koszul residue map, the induced Laplacian is (6.89). The filtered map is therefore a quasi-isomorphism, proving (6.90). Finally, the anomaly calculation in Section 6.6 identifies the localized current with the Virasoro current at central charge one. □

Remark 6.18 (The half-Ω localization). Let $S = \mathrm{T}^*\Sigma$, let v be the fiberwise Euler field, and let $q = \mathbb{1}_C \times p\colon C \times \mathrm{T}^*\Sigma \to C \times \Sigma$. The same argument, using (6.29) and (6.30), identifies

$$q_*\left(\widetilde{\mathrm{Obs}}^{v}\right)$$

with the free BV quantization of the degenerate Poisson theory on $C \times \Sigma$ whose fields are

$$\Omega^{0,\bullet}\left(C \times \Sigma, \Omega^1_C \boxtimes (\mathcal{O}_\Sigma \oplus \Omega^1_\Sigma[1])\right)$$

and whose Poisson operator is the off-diagonal ∂_C in (6.30). Locally, if the first field is written as $f = \partial_C \mathcal{g}$, the nonlocal action becomes

$$\int_{C\times\Sigma} \mathcal{b}\, \overline{\partial} \mathcal{g},$$

so this is the holomorphic $\beta\gamma$ system on $C \times \Sigma$. Thus, $q_* \left(\widetilde{\mathrm{Obs}}^v\right)$ is equivalent to the factorization algebra of observables of the holomorphic twist of the four-dimensional $\mathcal{N} = 1$ free chiral multiplet theory.

6.8 Extracting vertex algebras on the complex line

There is a close relationship between holomorphic factorization algebras on $C = \mathbf{C}$ and vertex algebras. Indeed, in the model of factorization algebras that we use, there are some mild technical conditions on a holomorphic factorization algebra which guarantee that its cohomology is endowed with the structure of a vertex algebra.

The required data and structure on a two-dimensional factorization algebra are as follows:

(a) First, $\mathcal{F}$ is holomorphically translation invariant. This means that $\mathcal{F}$ is translation invariant and in addition there is a homotopy $\overline{\eta}$ for the infinitesimal action by the anti-holomorphic derivative $\frac{\partial}{\partial \overline{z}}$.

(b) The factorization algebra is equipped with an S^1 action, where $S^1 = U(1)$ acts on $\mathbf{C}$ by rotations. We assume that the action of S^1 on $\mathcal{F}(D)$, for any disk D centered at zero, is *tame* in the sense that it extends to an action of the algebra of distributions on S^1. See [CG17, definition 2.1.2].

(c) Let $\mathcal{F}^{(k)}(D) \subset \mathcal{F}(D)$ be the weight k eigenspace of the S^1 action. We assume that for any bigger disk D' that the natural map $\mathcal{F}^{(k)}(D) \to \mathcal{F}^{(k)}(D')$ is a quasi-isomorphism and for k large enough the complex $\mathcal{F}^{(k)}(D)$ is contractible. Finally, for each k and disk D centered at zero we require that the cohomology of $\mathcal{F}^{(k)}(D)$ can be expressed as a countable sequential limit of finite-dimensional graded vector spaces.

Theorem 6.19 ([CG17]). *Suppose that $\mathcal{F}$ is a factorization algebra on $\mathbf{C}$ equipped with the data and structure outlined in the items above. Then the factorization product endows the graded vector space*

$$\mathbb{V}(\mathcal{F}) \overset{\text{def}}{=} \oplus_{k\in\mathbf{Z}} H^\bullet(\mathcal{F}_k(D)) \tag{6.91}$$

with the structure of a vertex algebra.

The preceding theorem gives the free-field example promised above.

Corollary 6.20. *For $C = \mathbf{C}$, the vertex algebra extracted from the localized free theory is the rank-one Heisenberg vertex algebra:*

$$\mathbb{V}\left(\mathrm{Loc}_{\pi,\mathsf{Eu}}(\widetilde{\mathrm{Obs}})\right) \cong \mathsf{Heis}. \tag{6.92}$$

Its standard conformal vector has central charge one.

Proof. The chiral boson factorization algebra satisfies the hypotheses of the preceding theorem, and the resulting vertex algebra is the Heisenberg vertex algebra. The conformal statement follows from the final assertion of Theorem 6.17. □

Moreover, since the Virasoro factorization algebra satisfies these conditions and yields the standard Virasoro vertex algebra, we have the following.

Proposition 6.21 ([Wil17b])**.** *Suppose that $\mathcal{F}$ is a chiral conformal factorization algebra on $\mathbf{C}$ of central charge c equipped with the data and structure outlined in the items above. Then $\mathbb{V}(\mathcal{F})$ is a conformal vertex algebra of central charge c.*

A *tame* Dolbeault conformal factorization algebra is a holomorphically translation invariant factorization algebra on $\mathbf{C}^3 = \mathbf{C} \times \mathbf{C}^2$ equipped with an S^1 action which rotates the first factor, such that $\mathrm{Loc}_{\pi,\mathsf{Eu}}(\mathcal{F})$ satisfies the items in the list above.

The Dolbeault factorization algebra $\mathbf{U}_{c\widetilde{\Theta}_{fr}}\left(\widetilde{\mathcal{E}}(3|6)\right)$ on $\mathbf{C}^3$ is tame. In fact, we have just shown that its localization is precisely the Virasoro factorization algebra. Thus we obtain the following.

Theorem 6.22. *Suppose that $\mathcal{F}$ is a tame Dolbeault factorization algebra on $\mathbf{C}^3 = \mathbf{C} \times \mathbf{C}^2$ of Dolbeault central charge c. Then the factorization products endow $\mathbb{V}\left(\mathrm{Loc}_{\pi,\mathsf{Eu}}(\mathcal{F})\right)$ with the structure of a conformal vertex algebra of central charge c.*

This theorem provides a construction of a conformal vertex algebra associated to a six-dimensional $\mathcal{N} = (2,0)$ superconformal theory. Indeed, consider the factorization algebra of observables $\widetilde{\mathcal{F}}_{suco}$ associated to a superconformal theory on $\mathbf{R}^6$. We can then consider the polarized holomorphic twist of this factorization algebra which results in a new factorization algebra on $\mathbf{C}^3 = \mathbf{R}^6$:

$$\mathcal{F}_{susy} \rightsquigarrow \mathcal{F} = \left(\widetilde{\mathcal{F}}_{susy}, Q_{susy} + Q_{\mathrm{hol}}\right) \tag{6.93}$$

Here, $\widetilde{\mathcal{F}}_{susy}$ denotes the regraded and twisted factorization algebra, following the polarized twisting data. The differential Q_{susy} is the original BRST differential of the supersymmetric theory, and Q_{hol} is the holomorphic twisting supercharge. By theorem 3.1, which we recollected from [Hah+24], the factorization algebra $\mathcal{F}$ admits a classical background for the local Lie algebra $\widetilde{\mathcal{E}}(3|6)$. We then apply the theorem above to obtain a conformal vertex algebra structure on $\mathbb{V}\left(\mathrm{Loc}_{\pi,\mathsf{Eu}}(\mathcal{F})\right)$.

A From four-dimensional $\mathcal{N} = 2$ theories to vertex algebras

In the main text, we constructed a functor from six-dimensional $E(3|6)$-structured factorization algebras to vertex algebras. Such factorization algebras arise naturally by twisting theories with six-dimensional superconformal symmetry. There is a parallel construction in four dimensions, inspired by the correspondence of [Bee+15], which associates a vertex algebra to a four-dimensional $\mathcal{N} = 2$ superconformal theory. In [SW23b], we formulated this four-dimensional construction in terms of factorization algebras and the holomorphic twist.

This appendix recalls the construction in a form parallel to the $E(3|6)$ story. We work throughout with $\mathbb{Z}$-graded objects. The grading refines the parity grading that one obtains from the holomorphic twist; reducing it modulo two recovers the corresponding Lie superalgebra.

A.1 A graded extension of holomorphic vector fields on $\mathbf{C}^2$

Write

$$\mathbf{C}^2 = \mathbf{C}_z \times \mathbf{C}_w$$

and, whenever the indicated powers have been chosen, set

$$K^{r,s} = K_{\mathbf{C}_z}^{\otimes r} \boxtimes K_{\mathbf{C}_w}^{\otimes s}.$$

Thus a local section of $K^{r,s}$ has the form $f(z,w)(\mathrm{d}z)^r(\mathrm{d}w)^s$.

Let $\widehat{\Gamma}_0(-)$ denote formal sections at the origin. Let $\mathfrak{v}_{r,s}$ denote the $\mathbb{Z}$-graded Lie algebra with underlying graded vector space

$$\mathfrak{v}_{r,s}^{-1} = \widehat{\Gamma}_0\big(\mathrm{T}_{\mathbf{C}^2} \otimes K^{r,s}\big), \tag{A.1}$$

$$\mathfrak{v}_{r,s}^{0} = \mathrm{Der}(\mathbf{C}[\![z,w]\!]) \ltimes \mathbf{C}[\![z,w]\!], \tag{A.2}$$

$$\mathfrak{v}_{r,s}^{1} = \widehat{\Gamma}_0\big(K^{-r,-s}\big). \tag{A.3}$$

Its bracket is the one induced by vector fields on the parity-shifted line bundle $\Pi K^{-r,-s}$. This description fixes the signs while allowing us to retain the full $\mathbb{Z}$-grading rather than only its reduction modulo two. After trivializing the canonical bundles by $\mathrm{d}z$ and $\mathrm{d}w$, the graded Lie algebras $\mathfrak{v}_{r,s}$ are isomorphic. We retain (r,s) because geometrically natural representations depend on these weights.

The corresponding local dg Lie algebra is the Dolbeault resolution whose internal graded pieces are

$$\begin{aligned}
\mathscr{L}_{r,s}^{-1} &= \Omega^{0,\bullet}(\mathbf{C}^2, \mathrm{T}_{\mathbf{C}^2} \otimes K^{r,s}),\\
\mathscr{L}_{r,s}^{0} &= \Omega^{0,\bullet}(\mathbf{C}^2, \mathrm{T}_{\mathbf{C}^2} \ltimes \mathscr{O}_{\mathbf{C}^2}),\\
\mathscr{L}_{r,s}^{1} &= \Omega^{0,\bullet}(\mathbf{C}^2, K^{-r,-s}).
\end{aligned}$$

Its total cohomological degree is the sum of the Dolbeault degree and the displayed internal degree. The differential is $\overline{\partial}$, and the bracket is induced from $\mathfrak{v}_{r,s}$. Taking formal jets at the origin recovers $\mathfrak{v}_{r,s}$.

Let $\Theta \in H^1_{\mathrm{loc}}(\mathscr{L}_{r,s})$ be a local cocycle. We say that a factorization algebra $\mathscr{F}$ on $\mathbf{C}^2$ has an $\mathscr{L}_{r,s}$-structure with anomaly Θ if it is equipped with a map of factorization algebras

$$\mathbf{T}_{r,s} \colon \mathbf{U}_\Theta(\mathscr{L}_{r,s}) \longrightarrow \mathscr{F}. \tag{A.4}$$

Here $\mathbf{U}_\Theta(\mathscr{L}_{r,s})$ denotes the quantum factorization envelope with central extension determined by Θ. The map $\mathsf{T}_{r,s}$ is the holomorphic stress tensor for this graded symmetry.

A.2 Localization to the z-plane

We now set $r = 0$ and abbreviate

$$\mathscr{L}_s = \mathscr{L}_{0,s}.$$

Let

$$\pi\colon \mathbf{C}_z \times \mathbf{C}_w \longrightarrow \mathbf{C}_z$$

be the projection. The degree-one element

$$u = w(\mathrm{d}w)^{-s} \in \Gamma(\mathbf{C}^2, K^{0,-s}) \subset \mathscr{L}_s^1 \tag{A.5}$$

is holomorphic and satisfies $[u_s, u_s] = 0$, hence is a Maurer–Cartan element. Denote the corresponding deformation by

$$\mathscr{L}_s^u = \big(\mathscr{L}_s, \bar\partial + [u, -]\big).$$

Suppose that $\mathscr{F}$ is equipped with an $\mathscr{L}_s$-structure $\mathbf{T}_s$. Its localization to the z-plane is the factorization algebra

$$\mathrm{Loc}_u(\mathscr{F}) = (\pi_*\mathscr{F}, \mathrm{d}_{\mathscr{F}} + \mathbf{T}_s(u)) \,. \tag{A.6}$$

The factorization products are inherited from $\pi_*\mathscr{F}$. The notation in (A.6) emphasizes that u deforms $\mathscr{F}$ through the action $\mathbf{T}_s$; it is not added directly to the differential of $\mathscr{F}$.

The basic local calculation of [SW23b] gives a quasi-isomorphism

$$\pi_*\mathscr{L}_s^u \simeq \mathscr{T}_{\mathbf{C}_z},$$

where

$$\mathscr{T}_{\mathbf{C}_z} = \Omega^{0,\bullet}(\mathbf{C}_z, \mathrm{T}_{\mathbf{C}_z})$$

is the local dg Lie algebra of holomorphic vector fields. Consequently, $\mathrm{Loc}_u(\mathscr{F})$ carries a holomorphic stress tensor. It remains to determine its Virasoro anomaly.

A.3 The case $s = 0$

For the anomaly computation we take $s = 0$ and write

$$\mathscr{L} = \mathscr{L}_{0,0}, \qquad \mathfrak{v} = \mathfrak{v}_{0,0}, \qquad S = S_0 = w.$$

Using descent, see [Wil24b], there is the following relationship between the local Lie algebra cohomology of $\mathscr{L}$ and the Gelfand–Fuks Lie algebra cohomology of $\mathfrak{v}$:

$$H^\bullet_{\mathrm{loc}}\big(\mathscr{L}(\mathbf{C}^2)\big) \cong H^\bullet_{\mathrm{GF}}(\mathfrak{v})[4]. \tag{A.7}$$

Notice that $\mathfrak{v}$ is a regraded version of the Lie algebra of supervector fields on the $2|1$ super space. The Gelfand–Fuks cohomology of $\mathfrak{v}$ was computed in [Pim26]. In degree five (so local degree 1) there is an identification with universal Chern classes:

$$H^1_{\mathrm{loc}}\big(\mathscr{L}(\mathbf{C}^2)\big) \cong H^5(\mathfrak{v}) \cong H^4(BU(2)), \tag{A.8}$$

In particular, the space of anomalies for a $\mathscr{L}$-symmetry on $\mathbf{C}^2$ is two-dimensional.

The degree-zero local Lie algebra is

$$\mathscr{L}^0 = \Omega^{0,\bullet}\big(\mathbf{C}^2, \mathrm{T}_{\mathbf{C}^2} \ltimes \mathscr{O}_{\mathbf{C}^2}\big).$$

The same ideas presented in [Wil24b] identify its relevant local cohomology with $H^6(BU(2)\times BU(1))$. Restriction along $\mathscr{L}^0 \hookrightarrow \mathscr{L}$ therefore gives

$$H^1_{\mathrm{loc}}(\mathscr{L}) \longrightarrow H^1_{\mathrm{loc}}(\mathscr{L}^0) \cong H^6(BU(2) \times BU(1)).$$

Let c_1, c_2 denote the universal Chern classes of $U(2)$ and let $\widetilde{c}_1$ denote the generator of $H^2(BU(1))$. Write

$$\mathrm{ch}_1 = c_1, \qquad \mathrm{ch}_2 = \frac{1}{2}(c_1^2 - 2c_2).$$

Under the preceding restriction, the anomaly with parameters $(\alpha, \beta) \in \mathbf{C}^2$ is represented by

$$\Theta_{\alpha,\beta} = \alpha\, \mathrm{ch}_1^2\, \widetilde{c}_1 + \beta\, \mathrm{ch}_2\, \widetilde{c}_1 + 2\alpha\, \mathrm{ch}_1\, \widetilde{c}_1^2 + 2(\alpha - 2\beta)\widetilde{c}_1^3. \tag{A.9}$$

Theorem A.1. *Let $\mathscr{F}$ be a factorization algebra on $\mathbf{C}^2$ equipped with an $\mathscr{L}$-structure of anomaly $\Theta_{\alpha,\beta}$. Then $\mathrm{Loc}_S(\mathscr{F})$ is a chiral conformal factorization algebra on $\mathbf{C}_z$ with central charge*

$$c_{2d} = -24\left(\alpha + \frac{1}{2}\beta\right). \tag{A.10}$$

The theorem follows from the corresponding statement for the universal factorization envelope.

Proposition A.2. *There is an equivalence of factorization algebras*

$$\mathrm{Loc}_S\big(\mathbf{U}_{\Theta_{\alpha,\beta}}(\mathscr{L})\big) \simeq \mathbf{U}_{-(\alpha+\frac{1}{2}\beta)\Theta_{\mathrm{Vir}}}(\mathscr{T}_{\mathbf{C}_z}). \tag{A.11}$$

With the convention that the Virasoro cocycle Θ_{Vir} has central charge 24, the right-hand side has central charge (A.10).

Proof. For every open set $U \subset \mathbf{C}_z$, [SW23b] constructs an L_∞ quasi-isomorphism

$$\Phi\colon \Omega^{0,\bullet}_c(U, \mathrm{T}_{\mathbf{C}_z}) \rightsquigarrow (\mathscr{L}^S)_c(U \times \mathbf{C}_w). \tag{A.12}$$

Choose a smooth compactly supported function ρ on $\mathbf{C}_w$ that is equal to one near $w = 0$. The linear term of Φ is

$$\Phi^{(1)}(\mu_z) = \left(\rho - \varepsilon\frac{\overline{\partial}\rho}{w}\right)\mu_z. \tag{A.13}$$

Here ε is a bookkeeping variable of degree -1: the second summand in (A.13) lies in the internal degree -1 summand $\mathscr{L}^{-1} = \Omega^{0,\bullet}(\mathbf{C}^2, \mathrm{T}_{\mathbf{C}^2})$.

We use the following notation for inputs of $\mathscr{L}$. Write (μ, A) for the vector-field and scalar inputs in degree zero, write μ' for an input in degree -1, and write A' for an input in degree $+1$. Let $J\mu$ denote the holomorphic Jacobian of μ and let ∂ denote the holomorphic de Rham differential.

The cocycle on the deformed algebra is $e^{\iota_S}\Theta_{\alpha,\beta}$. Upon restriction along Φ, only terms containing one μ, one μ', and one insertion $A' = S$ can contribute. Thus only the following parts of the cocycle representatives are relevant:

$$(2\pi\mathrm{i})^2\,\mathrm{ch}_1^2\,\widetilde{c}_1 \rightsquigarrow \int \mathrm{Tr}(J\mu)\,\partial\,\mathrm{Tr}(J\mu')\,\partial A', \tag{A.14}$$

$$(2\pi\mathrm{i})^2\,\mathrm{ch}_2\,\widetilde{c}_1 \rightsquigarrow \frac{1}{2}\int \mathrm{Tr}(J\mu\,\partial J\mu')\,\partial A'. \tag{A.15}$$

The terms proportional to $\mathrm{ch}_1\,\widetilde{c}_1^2$ and $\widetilde{c}_1^3$ vanish after restriction along Φ.

Substituting (A.13) and $A' = S = w$ into (A.14), the contribution of the first anomaly parameter is

$$\begin{aligned} -\frac{\alpha}{(2\pi\mathrm{i})^2}\int_{U\times\mathbf{C}_w} &\mathrm{Tr}\big(J(\rho\mu_z)\big)\,\partial\,\mathrm{Tr}\left(J\left(\frac{\overline{\partial}\rho}{w}\mu_z\right)\right)\mathrm{d}w \\ &= -\frac{\alpha}{(2\pi\mathrm{i})^2}\int_{U\times\mathbf{C}_w}(J\mu_z\,\partial J\mu_z)\,\overline{\partial}(\rho^2)\frac{\mathrm{d}w}{w} \\ &= -\frac{\alpha}{2\pi\mathrm{i}}\int_U J\mu_z\,\partial J\mu_z. \end{aligned}$$

Similarly, (A.15) contributes

$$-\frac{\beta/2}{2\pi\mathrm{i}}\int_U J\mu_z\,\partial J\mu_z.$$

Therefore

$$\Phi^*\left(e^{\iota_S}\Theta_{\alpha,\beta}\right) = -\left(\alpha + \frac{1}{2}\beta\right)\frac{1}{2\pi\mathrm{i}}\int_U J\mu_z\,\partial J\mu_z. \tag{A.16}$$

Our normalization of the Virasoro anomaly is

$$\frac{c_{2d}}{24}\,\frac{1}{2\pi\mathrm{i}}\int_U J\mu_z\,\partial J\mu_z.$$

Comparing with (A.16) gives $c_{2d} = -24(\alpha + \frac{1}{2}\beta)$ and proves the proposition. $\square$

At the level of the terms that survive localization, the calculation can be viewed as the specialization $\widetilde{c}_1 = -1$ of the four-dimensional anomaly polynomial. It would be useful to give a direct geometric or Lie-algebraic explanation of this specialization.

A.4 A basic example

As a simple example, consider the factorization algebra associated to the free $\beta\gamma$-system on $\mathbf{C}^{2|1}$. The fields of this model are

$$\begin{aligned} \Gamma &\in \Omega^{0,\bullet}(\mathbf{C}^{2|1}) = \Omega^{0,\bullet}(\mathbf{C}^2)[\epsilon] \\ B &\in \Omega^{2,\bullet}(\mathbf{C}^{2|1}) = \Omega^{2,\bullet}(\mathbf{C}^2)[\epsilon][1] \end{aligned}$$

and the action is $\int_{\mathbf{C}^{2|1}} B\bar{\partial}\Gamma$, where the Berezinian integration over $\mathbf{C}^{2|1}$ means that we pick up the geometric top form of Dolbeault type $(2,2)$ and the coefficient of ϵ.

The $\mathfrak{vect}(2|1)$-equivariant structure on this theory is universal. Explicitly, the coupling involving a graded vector field

$$\mathbf{X} \in \Omega^{0,\bullet}(\mathbf{C}^{2|1}, \mathrm{T}_{\mathbf{C}^{2|1}}) \tag{A.17}$$

is given by

$$\mathbf{T}_{2|1}(\mathbf{X}) = \int_{\mathbf{C}^{2|1}} BL_{\mathbf{X}}\Gamma, \tag{A.18}$$

where $L_{\mathbf{X}}$ denotes the (super) Lie derivative.

The $\mathfrak{vect}(2|1)$-anomaly polynomial $\Theta_{\alpha,\beta}$, in this example, has

$$\alpha = -\frac{1}{8}, \quad \beta = \frac{1}{12}. \tag{A.19}$$

This can be computed by explicit Feynman diagrams or by invoking the Riemann–Roch theorem [Wil24a].

The localization of the factorization algebra associated to this free theory on $\mathbf{C}^2$ is the $\beta\gamma$ system on $\mathbf{C}$ whose fields are simply

$$\begin{aligned} \gamma &\in \Omega^{0,\bullet}(\mathbf{C}) \\ \beta &\in \Omega^{1,\bullet}(\mathbf{C}), \end{aligned}$$

and action $\int \beta\bar{\partial}\gamma$. The associated vertex algebra is the $\beta\gamma$ vertex algebra where γ has spin zero and β is spin 1. The central charge of this vertex algebra is $c_{2d} = 2$, which we note agrees with the general formula

$$c_{2d} = -24\left(-\frac{1}{8} + \frac{1}{24}\right) = 2, \tag{A.20}$$

of theorem A.1.

A.5 Examples from supersymmetry

An important source of examples of factorization algebras with $\mathfrak{vect}(2|1)$ structures arise from four-dimensional superconformal symmetry. Indeed, the four-dimensional $\mathcal{N} = 2$ Weyl multiplet admits a holomorphic twist which renders it equivalent to the local Lie algebra $\mathcal{L}_{2|1}$, [Hah+24].

Just like in the six-dimensional case, there is some freedom in what one means by a "holomorphic twist". There is a one-parameter family of twisting homomorphism which are compatible with the holomorphic twisting supercharge. For example, if we start with a theory equipped with a $\mathcal{N} = 2$ superconformal structure, then it's r-holomorphic twist is a holomorphic theory on $\mathbf{C}^2$ which has a symmetry by the infinite-dimensional Lie superalgebra $\mathfrak{vect}(\Pi K^r)$. When $r = 0$, this corresponds to simply $\mathfrak{vect}(2|1)$ (the corresponding local Lie algebra was denoted $\mathcal{L}_{2|1}$); this is the case which is relevant for extracting vertex algebras via the approach pioneered in [Bee+15].

Recall that the space of $\mathscr{L}_{2|1}$-anomalies is two-dimensional. We described this space using the coefficients α, β in equation (A.9). The space of anomalies for conformal symmetry is also two dimensional, typically described in terms of the coefficients a, c of the Euler and Weyl tensors, respectively. Equivalently, these are the two parameters which appear in the central terms of the four-dimensional TT OPE [Bee+15]. Suppose that we start with a $\mathcal{N} = 2$ superconformal theory whose conformal anomalies are a, c. The holomorphic twist of this theory is $\mathscr{L}_{2|1}$-equivariant. It turns out that the $\mathscr{L}_{2|1}$-anomaly coefficients α, β are completely determined by the coefficients a, c of the original untwisted superconformal theory. The exact relation depends on the twisting homomorphism used to describe the holomorphic twist. When $r = 0$, which is the twist which leads to the construction of vertex algebras, there is the following linear relations among coefficients:

$$
\begin{aligned}
\alpha &= \frac{1}{2}(2a - c) \\
\beta &= 2(c - a).
\end{aligned}
$$

Consider a simple example of a free hypermultiplet. Recall, as a $\mathcal{N} = 1$ multiplet the free hypermultiplet is equivalent to two decoupled chiral multiplets. Being superconformal forces the charge under the R-symmetry $G_{R_{\mathcal{N}=1}} = U(1)_{R_{\mathcal{N}=1}}$ to be $2/3$. As an $\mathcal{N} = 1$ superconformal field theory, the holomorphic twist of a free hypermultiplet is the $\mathfrak{vect}(2)$-equivariant theory whose fields are

$$
\begin{aligned}
\gamma = (\gamma^i) &\in \Omega^{0,\bullet}(\mathbf{C}^2, K_{\mathbf{C}^2}^{1/3}) \otimes \mathbf{C}^2 \\
\beta = (\beta_j) &\in \Omega^{0,\bullet}(\mathbf{C}^2, K_{\mathbf{C}^2}^{2/3}) \otimes \mathbf{C}^2[1],
\end{aligned}
$$

where $i = 1, 2$. The action functional is

$$
\int_{\mathbf{C}^2} \beta_i \bar{\partial} \gamma^i = \int_{\mathbf{C}^2} \beta_1 \bar{\partial} \gamma^1 + \int_{\mathbf{C}^2} \beta_2 \bar{\partial} \gamma^2. \tag{A.21}
$$

In terms of our parametrization, this holomorphic twist corresponds to $r = 1/3$, meaning the Lie superalgebra $\mathfrak{vect}(\Pi K^{1/3})$ is naturally a symmetry of this free holomorphic theory.

If we instead use the twisting data corresponding the value $r = 0$, the twist of a free hypermultiplet is the $\mathfrak{vect}(2|1)$-equivariant theory whose fields are

$$
\Phi = (\Phi_i) = (\varphi_i + \epsilon \varphi'_i) \in \Omega^{0,\bullet}(\mathbf{C}^2, K_{\mathbf{C}^2}^{1/2}) \otimes \mathbf{C}^2[\epsilon], \tag{A.22}
$$

where ϵ carries degree -1. The action functional is

$$
\frac{1}{2} \int_{\mathbf{C}^{2|1}} \varepsilon^{ij} \Phi_i \bar{\partial} \Phi_j = \int_{\mathbf{C}^2} \varepsilon^{ij} \varphi_i \bar{\partial} \varphi'_j. \tag{A.23}
$$

The conformal anomaly coefficients of the fully supersymmetric theory are known to be $a = 1/24$ and $c = 1/12$. From our relation, we see that $\alpha = 0, \beta = 1/12$ for the holomorphic twist of a $\mathcal{N} = 2$ superconformal hypermultiplet. This is compatible with the computation of the $\mathfrak{vect}(2|1)$ anomaly computed directly from the description (A.23). Following the exact arguments in [SW23b], one sees that the localization obtained by deforming by $\mathrm{S} = w\frac{\partial}{\partial\epsilon}$ of this theory to one complex dimension is the free chiral CFT with fields

$$\phi \in \Omega^{0,\bullet}(\mathbf{C}, K_{\mathbf{C}}^{1/2}) \otimes \mathbf{C}^2 \tag{A.24}$$

with action $\int_{\mathbf{C}} \epsilon^{ij}\phi_i\bar{\partial}\phi_j$. This is the free CFT whose associated vertex algebra has two generating fields $\phi_1(z), \phi_2(z)$ each of spin 1/2 and OPE

$$\phi_i(z)\phi_j(w) \simeq \frac{\varepsilon_{ij}}{z-w}. \tag{A.25}$$

This is referred to in the literature as the 'free symplectic boson' chiral CFT. Its central charge is $c_{2d} = -1$ as expected from the formula (A.1).

More generally, the hypermultiplet may take values in a symplectic vector space U (which above is simply $U = \mathbf{C}^2$). The localized vertex algebra is the chiral symplectic boson vertex algebra valued in U of central charge $c = -\frac{1}{2}\dim U$.

We next consider the example of a vector multiplet. In other words, pure four-dimensional $\mathcal{N} = 2$ Yang–Mills theory associated to G, whose Lie algebra we denote by $\mathfrak{g}$.

Viewed as an $\mathcal{N} = 1$ superconformal theory, the holomorphic twist of the free vector has fields

$$\Omega^{0,\bullet}(\mathbf{C}^2, \mathfrak{g})[1] \oplus \Omega^{0,\bullet}(\mathbf{C}^2, K_{\mathbf{C}^2}^{1/3}, \mathfrak{g})$$
$$\Omega^{0,\bullet}(\mathbf{C}^2, K_{\mathbf{C}^2}^{2/3}, \mathfrak{g}^*)[1] \oplus \Omega^{2,\bullet}(\mathbf{C}^2, \mathfrak{g}^*).$$

Again, this corresponds to the twisting parameter $r = 1/3$ in our convention. Viewed as an $\mathcal{N} = 2$ superconformal theory, the holomorphic twist of the free vector has fields

$$\Omega^{0,\bullet}(\mathbf{C}^2, \mathfrak{g})[\epsilon][1]$$
$$\Omega^{2,\bullet}(\mathbf{C}^2, \mathfrak{g}^*)[\epsilon][1].$$

This corresponds to $r = 0$. We consider this case now.

From the Riemann–Roch theorem we can read off the anomaly polynomial (A.9) associated to this $\mathcal{L}_{2|1}$ equivariant theory. The values of α, β are

$$\alpha = \frac{1}{8}\dim\mathfrak{g}, \quad \beta = -\frac{1}{12}\dim\mathfrak{g}. \tag{A.26}$$

In particular, the central charge of the localized vertex algebra is $c_{2d} = -2\dim\mathfrak{g}$. More generally, we can consider the twist of supersymmetric Yang–Mills theory with Lie algebra $\mathfrak{g}$ and where the hypermultiplets are valued in the symplectic $\mathfrak{g}$-representation U. The central charge of the corresponding vertex algebra is $c_{2d} = -2\dim\mathfrak{g} - \frac{1}{2}\dim U$.

We can check that this central charge is correct by directly characterizing the localized vertex algebra corresponding to $\mathcal{N} = 2$ supersymmetric Yang–Mills. It is a dg vertex algebra given by the BRST reduction of the chiral symplectic boson vertex algebra valued in the $\mathfrak{g}$-representation U. Thus, its structure as an underlying graded vertex algebra is a tensor product of the bc system valued in $\mathfrak{g}$ with the chiral boson vertex algebra valued in U:

$$bc\,[\mathfrak{g}] \otimes \mathrm{Symp}[U]. \tag{A.27}$$

References

[AS14] A. Altomani and A. Santi. "Classification of maximal transitive prolongations of super-Poincaré algebras." *Advances in Mathematics* 265 (2014), pp. 60–96. arXiv: 1212.1826 [math.RA].

[BRR15] C. Beem, L. Rastelli, and B. C. van Rees. "$\mathcal{W}$ symmetry in six dimensions." *Journal of High-Energy Physics* 05 (2015), p. 017. arXiv: 1404.1079 [hep-th].

[Bee+15] C. Beem et al. "Infinite chiral symmetry in four dimensions." *Comm. Math. Phys.* 336.3 (2015), pp. 1359–1433.

[BY16] D. Butson and P. Yoo. "Degenerate classical field theories and boundary theories" (2016). arXiv: 1611.00311 [math-ph].

[CK99] S.-J. Cheng and V. Kac. "Structure of some $\mathbf{Z}$-graded Lie superalgebras of vector fields." *Transformation Groups* 4.2-3 (1999), pp. 219–272.

[Cos11] K. Costello. *Renormalization and effective field theory*. Vol. 170. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2011.

[CG17] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory*. Vol. 1. New Mathematical Monographs 31. Cambridge University Press, 2017.

[CG21] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory*. Vol. 2. New Mathematical Monographs 41. Cambridge University Press, 2021.

[ESW20] C. Elliott, P. Safronov, and B. Williams. "A taxonomy of twists of supersymmetric Yang–Mills theory" (2020). arXiv: 2002.10517 [math-ph].

[FHK19] G. Faonte, B. Hennion, and M. Kapranov. "Higher Kac–Moody algebras and moduli spaces of G-bundles." *Advances in Mathematics* 346 (2019), pp. 389–466.

[Gro96] I. Grojnowski. "Instantons and affine algebras I: the Hilbert scheme and vertex operators." *Math. Res. Lett.* 3.2 (1996), pp. 275–291. URL: https://doi-org.ezproxy.bu.edu/10.4310/MRL.1996.v3.n2.a12.

[GWW26] Z. Gui, M. Wang, and B. R. Williams. *Higher-dimensional Chiral Algebras in the Jouanolou Model and free-field realization*. 2026. arXiv: 2510.26608 [math.QA]. URL: https://arxiv.org/abs/2510.26608.

[GS64] V. W. Guillemin and S. Sternberg. "An algebraic model of transitive differential geometry." *Bulletin of the American Mathematical Society* 70.1 (1964), pp. 16–47.

[GRW22] O. Gwilliam, E. Rabinovich, and B. R. Williams. "Factorization algebras and abelian CS/WZW-type correspondences." *Pure Appl. Math. Q.* 18.4 (2022), pp. 1485–1553. URL: https://doi.org/10.4310/pamq.2022.v18.n4.a7.

[GW18] O. Gwilliam and B. R. Williams. "Higher Kac–Moody algebras and symmetries of holomorphic field theories" (2018). arXiv: 1810.06534 [math.QA].

[Hah+24] F. Hahner, S. Raghavendran, I. Saberi, and B. R. Williams. "Local superconformal algebras" (2024). arXiv: 2410.08176 [math-ph].

[Hin97] V. Hinich. "Homological algebra of homotopy algebras." *Comm. Algebra* 25.10 (1997), pp. 3291–3323. URL: https://doi.org/10.1080/00927879708826055.

[Kac98] V. G. Kac. "Classification of infinite-dimensional simple linearly compact Lie superalgebras." *Adv. Math.* 139.1 (1998), pp. 1–55. URL: https://doi.org/10.1006/aima.1998.1756.

[Kho06] A. S. Khoroshkin. "Lie algebra of formal vector fields extended by formal g-valued functions." *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 335 (2006), pp. 205–230. URL: https://doi-org.ezproxy.bu.edu/10.1007/s10958-007-0167-5.

[Knu18] B. Knudsen. "Higher enveloping algebras." *Geom. Topol.* 22.7 (2018), pp. 4013–4066. URL: https://doi-org.ezproxy.bu.edu/10.2140/gt.2018.22.4013.

[Lur11] J. Lurie. "Derived algebraic geometry X: formal moduli problems." 2011. URL: https://www.math.ias.edu/~lurie/papers/DAG-X.pdf.

[Mac87] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*. London Mathematical Society Lecture Notes 124. Cambridge University Press, 1987.

[Nah78] W. Nahm. "Supersymmetries and their representations." *Nuclear Physics B* 135 (1978), p. 149.

[Nak97] H. Nakajima. "Heisenberg algebra and Hilbert schemes of points on projective surfaces." *Ann. of Math. (2)* 145.2 (1997), pp. 379–388. URL: https://doi-org.ezproxy.bu.edu/10.2307/2951818.

[Pim26] S. Pimenov. "Gelfand-Fuchs cohomology for affine superspaces $\mathbb{A}^{m,n}$." *Beijing J. Pure Appl. Math.* 3.1 (2026), pp. 121–141. URL: https://doi-org.ezproxy.bu.edu/10.4310/bpam.251216170308.

[RSW23] S. Raghavendran, I. Saberi, and B. R. Williams. "Twisted eleven-dimensional supergravity." *Communications in Mathematical Physics* 402.2 (2023), pp. 1103–1166. arXiv: 2111.03049 [math-ph].

[SW23a] I. Saberi and B. R. Williams. "Constraints in the BV formalism: six-dimensional supersymmetry and its twists." *Adv. Math.* 412 (2023), Paper No. 108806, 87. URL: https://doi.org/10.1016/j.aim.2022.108806.

[SW23b] I. Saberi and B. R. Williams. "Superconformal algebras and holomorphic field theories." *Annales Henri Poincaré* 24.2 (2023), pp. 541–604. arXiv: 1910.04120 [math-ph].

[Shn88] S. Shnider. "The superconformal algebra in higher dimensions." *Letters in Mathematical Physics* 16 (1988), pp. 377–383.

[Wil17a] B. Williams. "The Virasoro vertex algebra and factorization algebras on Riemann surfaces." *Lett. Math. Phys.* 107.12 (2017), pp. 2189–2237. URL: https://doi.org/10.1007/s11005-017-0982-7.

[Wil17b] B. R. Williams. "The Virasoro vertex algebra and factorization algebras on Riemann surfaces." *Letters in Mathematical Physics* 107.12 (2017), pp. 2189–2237.

[Wil20] B. R. Williams. "Renormalization for holomorphic field theories." *Comm. Math. Phys.* 374.3 (2020), pp. 1693–1742. URL: https://doi.org/10.1007/s00220-020-03693-5.

[Wil24a] B. R. Williams. "A complex geometric perspective on the a and c anomalies" (2024). arXiv: 2404.16128 [math-ph].

[Wil24b] B. R. Williams. "The local cohomology of vector fields" (2024). arXiv: 2405.05174 [math.DG].

[Wil24c] B. R. Williams. "The local cohomology of vector fields" (2024). arXiv: 2405.05174 [math.DG].

[Wit97] E. Witten. "Fivebrane effective action in M-theory." *J. Geom. Phys.* 22 (1997), pp. 103–133. arXiv: 9610234 [hep-th].