FACTORIZATION ALGEBRAS FROM TOPOLOGICAL–HOLOMORPHIC FIELD THEORIES

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ABSTRACT. Topological field theories and holomorphic field theories naturally appear in both mathematics and physics. However, there exist intriguing hybrid theories that are topological in some directions and holomorphic in others, such as twists of supersymmetric field theories or Costello's 4-dimensional Chern-Simons theory. In this paper, we rigorously prove the ultraviolet (UV) finiteness for such hybrid theories on the model manifold $\mathbf{R}^{d'} \times \mathbf{C}^d$, and present two significant vanishing results regarding anomalies: in the case $d' = 1$, the odd-loop obstructions to quantization on $\mathbf{R}^{d'} \times \mathbf{C}^d$ vanish; in the case $d' > 1$, all obstructions disappear, allowing us to define a factorization algebra structure for quantum observables.

1. INTRODUCTION

This work continues the study of quantum field theories which exist on a product of manifolds of the form $M \times X$ where M is a smooth manifold and X is a complex manifold [GRW21; GKW24]. Such theories depend only on the smooth structure of the manifold M and only on the complex structure of the manifold X. In this sense, these quantum field theories generalize the well-studied *topological* field theories and *holomorphic* field theories which have garnered significant attention in recent years [CL15; Cos13a; Wil20; Wan24]. Examples of these hybrid theories appear as twists of supersymmetric theories, see for example [OY20; GRW23; Aga+17]. We recall a precise definition of a topological-holomorphic field theory in section 2.

The main result of this paper is that the perturbative quantization, on flat space $\mathbf{R}^{d'} \times \mathbf{C}^d$, of classical field theories which are of this hybrid topological and holomorphic type are particularly well-behaved. In particular, we will show that in many

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cases, quantizations always exist and can be produced diagrammatically using Feynman graph integrals. We refer to theorem 3.9 for a more detailed statement of this result.

Theorem 1.1. *The perturbative renormalization of a topological-holomorphic theory on* $\mathbf{R}^{d'} \times \mathbf{C}^d$ *is UV finite to all orders in perturbation theory.*

This theorem extends a result of the second author along with Gwilliam and Rabinovich to *all* orders in perturbation theory, rather than just to first-order [GRW21]. The main technique used here is the compactification of the space of Schwinger parameters, developed by the first author [Wan24]. Most of section 3 is dedicated to the proof of this result.

In the remainder of this introduction we detail a consequence of our main result which is a rigorous construction of factorization algebras associated to a wide variety of such topological-holomorphic theories.

Factorization algebras (and their cousins, chiral algebras) originally appeared in the work of Beilinson and Drinfeld [BD04] in their approach to codify algebraic structures in conformal field theory. More generally, as developed by Costello and Gwilliam, a factorization algebra encapsulates the observables in *any* quantum field theory [CG21]. To every open set $U \subset M$, the factorization algebra of observables Obs assigns assigns a complex vector space (this becomes a cochain complex in the Batalin–Vilkovisky formalism) that we denote $\mathrm{Obs}(U)$; this is the set of measurements that one can make in the patch of spacetime U. If $U \subset V \subset M$ is a nested sequence of open sets there is a extension map $\mathrm{Obs}(U) \to \mathrm{Obs}(V)$; this means that observables on U can be viewed as an observable on V. The most important structure arises from combining, or multiplying, observables. If U, U' are disjoint open sets which sit inside a larger open set V, then combining observables takes the form of a map $\star \colon \mathrm{Obs}(U) \otimes \mathrm{Obs}(U') \to \mathrm{Obs}(V)$. Iterating this, one obtains higher arity multiplication maps when there are more than two disjoint open sets involved, and there is an associativity axiom which relates such multiplications. Finally, there is a 'locality' axiom which relates the value of the factorization algebra on M (or some other open set) in terms of a cover of M by open sets. We refer to [CG17] for details.

In practice, we obtain such factorization algebras using perturbative renormalization, which is very close in spirit to deformation quantization. Suppose we have a classical field theory on a manifold M prescribed by some action functional. To every open set $U \subset M$, solutions to the Euler–Lagrange equations of the action functional restricted to U, taken up to gauge equivalences, defines a formal derived stack $\mathcal{EL}(U)$.

The factorization algebra of classical observables Obs^{cl} assigns to the open set U the algebra of functions on this derived stack $\mathrm{Obs}^{cl}(U) = \mathcal{O}(\mathcal{E}\mathcal{L}(U))$.

To study quantization we work in the formalism of quantum field theory developed in the series of books by Costello [Cos11] and Costello, Gwilliam [CG21]. We give a more detailed recollection in section 2. Perturbatively, a quantum field theory is defined over the ring $\mathbf{C}[[\hbar]]$. The problem is to find a factorization algebra Obs^q defined over $\mathbf{C}[[\hbar]]$ which reduces, modulo $\hbar$, to the factorization algebra of classical observables Obs^{cl}. Unlike deformation quantization, a complicating feature in field theory is that such quantizations may not exist. This is encapsulated in the form of *anomalies*. For a well-known instance of this we point out the Green–Schwarz mechanism which is used to cancel a one-loop anomaly in ten-dimensional supersymmetric Yang–Mills theory which can be cancelled by carefully coupling to ten-dimensional supergravity.

We prove anomalies vanish when a theory has at least two topological directions. We refer theorem 3.26 for a more detailed statement.

Theorem 1.2. *If $d' > 1$ then any classical perturbative topological-holomorphic theory defined on $\mathbf{R}^{d'} \times \mathbf{C}^d$ is anomaly free and admits a quantization to all orders in $\hbar$.*

In particular, any topological-holomorphic theory defines a factorization algebra of quantum observables Obs^q on $\mathbf{R}^{d'} \times \mathbf{C}^d$, defined over the ring $\mathbf{C}[[\hbar]]$, which satisfies

$$\mathrm{Obs}^q \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C} \cong \mathrm{Obs}^{cl} \tag{1}$$

with Obs^{cl} the factorization algebra of classical observables.

If the theory has only one topological direction, we show the odd-loop anomalies vanish. We refer proposition 3.23 for details.

Remark 1.3. There is a recent paper which shares some complementary overlap with our results [GKW24]. However, there are two differences:

(1) We prove the ultraviolet finiteness for all graphs, whereas [GKW24] focus only on an important class of graphs called Laman graphs. As far as we understand, finiteness for Laman graphs is not enough to prove the existence of the full quantum factorization algebra. However, by the conjectural formality for topological-holomorphic (colored) operads when $d' > 1$, the Feynman graph integrals for Laman graphs may contain all the information of the factorization algebra. In the approach of [GKW24] an explicit model for this operadic structure in the form of λ-brackets and descent is given and only depends on the behavior of Laman graphs.

(2) In the proof of anomaly freeness in [GKW24], they assumed an equivalence between topological gauge fixing and holomorphic gauge fixing, which doesn't seem obvious. We didn't assume such an equivalence.

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Conventions.

1. If $S_1, S_2, \ldots S_m$ are sets, $A_1, A_2, \ldots A_m$ are finite sets, an element in
$$\prod_{i=1}^{m} (S_i)^{|A_i|}$$
will be denoted by $(s_{a_1}, s_{a_2}, \ldots, s_{a_m})_{a_1 \in A_1, \ldots, a_m \in A_m}$, where s_{a_i} is an element in S_i.
2. Given a manifold M, we use $\Omega^\bullet(M)$ to denote the space of differential forms on M. The space of distributional valued differential forms is denoted by $\mathcal{D}^\bullet(M)$. If M is a complex manifold, to emphasize their bi-graded nature, we will use $\Omega^{\bullet,\bullet}(M)$ and $\mathcal{D}^{\bullet,\bullet}(M)$ for differential forms and distributional valued differential forms respectively.
3. We use $\widehat{\otimes}$ to denote the complete projective tensor product of nuclear vector spaces.
4. We use the following Koszul sign rules: If M, N are manifolds whose dimensions are m, n respectively, $\alpha \in \Omega^i(M), \beta \in \Omega^j(N)$
   i. Sign rule for differential forms:
   $$\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha.$$
   ii. Sign rule for integrations:
   $$\int_{M \times N} \alpha \wedge \beta = \int_M \int_N \alpha \wedge \beta = (-1)^{ni} \int_M \alpha \int_N \beta.$$

2. Batalin–Vilkovisky formalism for topological-holomorphic theories

We begin with an overview of the sort of classical field theories whose quantizations we consider in this paper. For more details we refer to [GRW21, section 1] [GKW24].

2.1. **Classical topological-holomorphic theories.** Throughout we will only work on $(d' + 2d)$-dimensional space $\mathbf{R}^{d'} \times \mathbf{C}^d$. The theories we consider are holomorphic 'along' $\mathbf{C}^d$ and topological 'along' $\mathbf{R}^{d'}$. In this short section we present such theories using action functionals, but the concept applies to theories without such a description.

For theories admitting a Lagrangian description, the action functional takes the form

$$S(\phi) = \int_{\mathbf{R}^{d'} \times \mathbf{C}^d} \mathrm{d}^d z \, \langle \phi, (\overline{\partial} + \mathrm{d})\phi \rangle + I(\phi) \tag{2}$$

where $\overline{\partial}$ is the $\overline{\partial}$-operator on $\mathbf{C}^d$ and d is the de Rham operator on $\mathbf{R}^{d'}$.[1] The interaction $I(\phi)$ is given by a Lagrangian density which depends only on the holomorphic jets of fields (meaning no derivatives along $\mathbf{R}^{d'}$ appear). More precisely,

$$I(\phi) = \sum_{k=3}^{\infty} \int_{\mathbf{R}^{d'} \times \mathbf{C}^d} \mathrm{d}^d z \, f_k \prod_{i=1}^{k} \partial^{\alpha_i} \phi,$$

where ∂^{α_i} are polynomials of holomorphic derivatives (constant coefficient holomorphic vector fields) and f_k are smooth differential forms.

We work within the Batalin–Vilkovisky formalism, meaning our space of fields is $\mathbf{Z}$-graded. In physics, this cohomological degree corresponds to the ghost number. The graded space of fields of a topological-holomorphic theory with action functional as above is of the form

$$\phi \in \Omega^\bullet(\mathbf{R}^{d'}) \,\widehat{\otimes}\, \Omega^{0,\bullet}(\mathbf{C}^d) \otimes V. \tag{3}$$

Here, V is a $\mathbf{Z}$-graded vector space equipped with a non-degenerate graded skew-symmetric pairing $\langle -, - \rangle_V$ of cohomological degree $d + d' - 1$. This pairing induces the BV pairing on compactly supported fields by integration against the holomorphic volume element

$$\omega(\phi, \phi') = \int_{\mathbf{R}^{d'} \times \mathbf{C}^d} \mathrm{d}^d z \, \langle \phi, \phi' \rangle_V. \tag{4}$$

Dual to this pairing is the so-called BV (anti) bracket which is defined on functionals of fields and is denoted $\{-, -\}$. In the BV formalism, the action functional of a classical

[1] In [GRW21] we consider the slightly more general situation where the kinetic term could also contain $\int \phi Q^{hol} \phi$ where Q^{hol} is some holomorphic differential operator. For our purposes, such a term can be absorbed in $I(\phi)$.

field theory is required to satisfy the classical master equation:

$$\{S, S\} = 0. \tag{5}$$

For a topological-holomorphic theory as in (2), this equation can equivalently be written as

$$\mathrm{d}I + \bar{\partial} I + \frac{1}{2}\{I, I\} = 0. \tag{6}$$

2.2. **Example: hybrid Chern–Simons theory.** In this subsection we survey a general class of topological-holomorphic theories which have appeared in recent work. Suppose that $\mathfrak{g}$ is a Lie algebra equipped with a non-degenerate, symmetric and invariant bilinear form $\langle -, - \rangle$. Suppose that $d' + d$ is a positive odd integer.

The space of fields of hybrid Chern–Simons theory is

$$\alpha \in \Omega^\bullet(\mathbf{R}^{d'}) \mathbin{\widehat{\otimes}} \Omega^{0,\bullet}(\mathbf{C}^d) \otimes \mathfrak{g}[1]. \tag{7}$$

The action functional is

$$S(\alpha) = \int_{\mathbf{R}^{d'} \times \mathbf{C}^d} \mathrm{d}^d z \left(\frac{1}{2}\langle \alpha, \mathrm{d}\alpha \rangle + \frac{1}{6}\langle \alpha, [\alpha, \alpha] \rangle \right). \tag{8}$$

Here, d denotes the de Rham differential in $d' + 2d$ dimensions, but notice that for type reasons only the component $\mathrm{d} + \bar{\partial}$ appears in the expression above. The fact that $\langle -, - \rangle$ is $\mathfrak{g}$-invariant implies that $\{S, S\} = 0$.

We recognize part of the integrand as the usual Chern–Simons Lagrangian. The appearance of the holomorphic volume element $\mathrm{d}^d z$ implies that only anti-holomorphic derivatives in $\mathbf{C}^d$ will appear in the kinetic term of the action.

The action functional carries cohomological degree $2(d' + d - 3)$, so that this theory is only $\mathbf{Z}$-graded when $d' + d = 3$. Otherwise, for general d, d' with $d' + d$ an odd integer this theory is merely $\mathbf{Z}/2$ graded (so that a field $\alpha \in \Omega^i(\mathbf{R}^{d'}) \mathbin{\widehat{\otimes}} \Omega^{0,j}(\mathbf{C}^d) \otimes \mathfrak{g}$ is of parity $(i + j - 1) \mod 2$).

We highlight a few other special cases.

(1) $d' = 3, d = 0$. This is ordinary (perturbative) topological Chern–Simons theory whose equations of motion describe deformations of the trivial flat G-bundle.
(2) $d' = 2, d = 1$. This is four-dimensional Chern–Simons theory as studied by Costello starting in [Cos13b] and further in [CWY18a; CWY18b]. Line operators in this theory are controlled by the Yangian quantum group.

(3) $d' = 1, d = 2$. When $\mathfrak{g} = \mathfrak{gl}_k$, there is a deformation of this theory which turns $\mathbf{C}^2$ non-commutative. The resulting 'non-commutative' Chern–Simons theory was studied by Costello in [Cos16] where it is shown that this theory arises from placing M-theory in the (twisted) Ω-background. Chiral surface operators in this theory encode the $W_{k+\infty}$-vertex algebra.
(4) $d' = 0, d = 5$. This is a purely holomorphic theory and has been shown in multiple sources to be the holomorphic twist of ten-dimensional supersymmetric Yang–Mills theory for the Lie algebra $\mathfrak{g}$ [Bau11; ESW22].
(5) $d' = 1, d = 4$. This is the twist of nine-dimensional supersymmetric Yang–Mills theory [ESW22].
(6) $d' = 2, d = 3$. This is the rank $(1,1)$ twist of eight-dimensional supersymmetric Yang–Mills theory [ESW22].
(7) $d' = 3, d = 2$. This is the twist of the rank 2 twist of seven-dimensional supersymmetric Yang–Mills theory [ESW22].
(8) $d' = 4, d = 1$. This is the rank $(2,2)$ twist of six-dimensional $\mathcal{N} = (1,1)$ supersymmetric Yang–Mills theory [ESW22].
(9) $d' = 5, d = 0$. This purely topological theory is the rank 4 twist of five-dimensional $\mathcal{N} = 2$ supersymmetric Yang–Mills theory [ESW22].

2.3. **Homotopical BV quantization.** In this section we swiftly recall the rigorous definition of a perturbative quantum field theory following Costello [Cos11] and Costello–Gwilliam [CG21]. For another great exposition we refer to [Li23]. We refer to these original references for more details.

We start with a classical BV theory on $\mathbf{R}^{d'} \times \mathbf{C}^d$, which for us is always of the form (2), but most of the discussion of this particular section is more general. Denote the space of fields by

$$\mathcal{E} \stackrel{\text{def}}{=} \Omega^\bullet(\mathbf{R}^{d'}) \otimes \Omega^{0,\bullet}(\mathbf{C}^d) \otimes V, \tag{9}$$

and let $Q = \mathrm{d} + \bar\partial$ for simplicity of notation. We use $\mathcal{E}^\vee$ to denote the dual space of $\mathcal{E}$ as topological vector space. The classical master equation is then

$$QI + \frac{1}{2}\{I, I\} = 0. \tag{10}$$

We remark that the BV bracket $\{-,-\}$ is not defined on all functionals of fields

$$\mathcal{O}(\mathcal{E}) = \prod_{n \geq 0} \mathrm{Sym}^n(\mathcal{E}^\vee) \tag{11}$$

as it involves contraction with the distributional form

$$K_0 \overset{\text{def}}{=} \omega^{-1} \in \mathrm{Sym}^2(\overline{\mathcal{E}})[-1] \tag{12}$$

with ω as in equation (4). The bar on the right hand side means we consider K_0 as a distributional section. The bracket is nevertheless well-defined on *local* functionals

$$\mathcal{O}_{loc}(\mathcal{E}) \tag{13}$$

which, by definition, are equivalence classes of Lagrangian densities where two densities are equivalent if they agree up to a total derivative [CG21]. For simplicity of exposition, we assume the interaction

$$I \in \mathcal{O}_{loc}(\mathcal{E}) \cap \mathcal{O}(\mathcal{E}),$$

i.e., I is a local functional with compact support. The compactness support assumption can be droped. See Remark 2.1.

The most important operator in the Batalin–Vilkovisky formalism it the BV Laplacian $\triangle$. This is naively defined as the contraction with the distributional kernel K_0, but just as with the BV bracket this is not well-defined on all functionals of the fields. The main idea involved with homotopical renormalization is to consider the new kernel

$$K_\Phi = K_0 - QP_\Phi \tag{14}$$

where

$$P_\Phi \overset{\text{def}}{=} \frac{1}{2} Q_{GF} \Phi \tag{15}$$

is the *propagator* associated to a *parametrix* Φ [CG21, §7.2]. Here Q^{GF} is a 'gauge fixing' operator for the theory, which for us is $Q^{GF} = \mathrm{d}^* + \overline{\partial}^*$. The key property of K_Φ is that it is a smooth (non-distributional) section of $\mathrm{Sym}^2(\mathcal{E})[-1]$ so that the operator $\triangle_\Phi$ defined by contraction with K_Φ is well-defined.

We will not recall the full definition of a parametrix as in [CG21], but we exhibit how they are used to define a quantum field theory. Below, we show how to produce parametrices using the heat equation. Following [CG21, definition 7.2.9.1], a ***quantum field theory*** is a family of functionals

$$\{I[\Phi]\} \subset \mathcal{O}(\mathcal{E})[[\hbar]], \tag{16}$$

one for each parametrix Φ, which satisfies a number of axioms. The two most important axioms are:

(1) *Renormalization group flow*. For every two parametrices Φ, Ψ the family must satisfy

$$I[\Phi] = W(P(\Phi) - P(\Psi), I[\Psi]) \tag{17}$$

where $W(-,-)$ is the Feynman diagram expansion (See Definition 2.4).

(2) *Quantum master equation*. For each Φ one has

$$(Q + \hbar \triangle_{\Phi}) e^{I[\Phi]/\hbar} = 0. \tag{18}$$

Obstructions to this equation holding are called ***anomalies***.

(3) *Classical limit*. If I is the interaction describing the classical field theory, then we say that $\{I[\Phi]\}$ is a quantization of I if $I = \lim_{\Phi \to 0} I[\Phi] \mod \hbar$.

We now point out how parametrices can be produced from the heat equation. Let

$$\Delta = [Q, Q^{GF}] = \bar{\partial} \circ \bar{\partial}^* + \bar{\partial}^* \circ \bar{\partial} + \mathrm{d} \circ \mathrm{d}^* + \mathrm{d}^* \circ \mathrm{d} \tag{19}$$

be the standard Laplacian on flat space $\mathbf{R}^{d'} \times \mathbf{C}^d$. Here, we recall that

$$\begin{cases} \bar{\partial}^* = -2 \sum_{i=1}^{d} \frac{\partial}{\partial z_i} i_{\frac{\partial}{\partial \bar{z}_i}} \\ \mathrm{d}^* = -\sum_{i=1}^{d'} \frac{\partial}{\partial x_i} i_{\frac{\partial}{\partial x_i}} \end{cases}$$

are the formal adjoints to the de Rham and Dolbeault operators with respect to the flat metric. This operator is not to be confused with the BV Laplacian $\triangle$ acting on functionals of fields.

We consider the associated heat kernel

$$H(t, z, x) = \frac{1}{2^{d+d'} (\pi t)^{d + \frac{d'}{2}}} e^{-\frac{2 \sum_{i=1}^{d} z_i \bar{z}_i + \sum_{i=1}^{d'} x_i^2}{4t}} \mathrm{d}^d \bar{z} \mathrm{d}^{d'} x, \quad t > 0 \tag{20}$$

This is a differential form solving the heat equation with initial condition

$$\begin{cases} \left(\frac{\partial}{\partial t} + \Delta \right) H(t, z, x) = 0 \\ H(t, z, x) \xrightarrow{t \to 0} \delta(z, x) \mathrm{d}^d \bar{z} \mathrm{d}^{d'} x, \end{cases} \tag{21}$$

with $\delta(z, x)$ the δ-distribution at $0 \in \mathbf{C}^d \times \mathbf{R}^{d'}$. Define

(22)

$$K_t(z - w, x - y) \stackrel{\text{def}}{=} H(t, z - w, x - y) \otimes \mathfrak{c}_V \in \mathcal{E}(\mathbf{R}^{d'} \times \mathbf{C}^d) \widehat{\otimes} \mathcal{E}(\mathbf{R}^{d'} \times \mathbf{C}^d) \widehat{\otimes} C^\infty(\mathbf{R}_+)$$

where $\mathfrak{c}_V$ is tensor dual to the pairing on V.[2] The parametrix associated to this heat kernel is

(23) $$\Phi_L \overset{\text{def}}{=} \int_0^L \mathrm{d}t\, K_t,$$

and the propagator is

(24) $$P_{0<L} = P(\Phi_L) = \int_{t=\varepsilon}^{L} (\bar{\partial}^* \otimes \mathbb{1} + \mathrm{d}^* \otimes \mathbb{1}) K_t.$$

Also, let $\triangle_L$ denote contraction with K_L and $P_{\epsilon<L} = P(\Phi_L) - P(\Phi_\epsilon)$. We use $\{-,-\}_L$ to denote the BV bracket defined by K_L.

Remark 2.1. If we use fake heat kernel instead of the usual heat kernel, we can drop the compact support assumption of the interaction (See [Cos11]). Our main results can be extended to this situation with little changes.

In what follows, we will produce quantizations from Feynman diagram expansions using propagators built explicitly from heat kernels. This means that we will produce a family of functionals $\{I[L]\}$ for each 'scale' $L > 0$ which satisfies the following key properties

(1) *Heat kernel renormalization group flow*. For every $\epsilon, L > 0$ the family must satisfy

(25) $$I[L] = W(P_{\epsilon<L}, I[\epsilon])$$

where $W(-,-)$ is the Feynman diagram expansion as above.

(2) *Heat kernel quantum master equation*. For each $L > 0$ one has

(26) $$(Q + \hbar\triangle_L)e^{I[L]/\hbar} = 0.$$

(3) *Heat kernel classical limit*. If I is the interaction describing the classical field theory, then we say that $\{I[L]\}$ is a quantization of I if $I = \lim_{L\to 0} I[L] \mod \hbar$.

For the additional axioms this family must satisfy we refer to [Cos11, definition 7.1.2], [CG21, definition 7.2.9.1]. We refer to such a family as a ***heat kernel quantum field theory***. Every heat kernel defines a parametrix Φ_L. Given a heat kernel quantum field theory, we obtain a quantum field theory in the sense above by the formula

(27) $$I[\Phi] = W(P(\Phi) - P(\Phi_L), I[L]).$$

[2]Fix a basis $\{v_i\}$ orthonormal for the pairing. Then $\mathfrak{c}_V = \sum_i v_i \otimes v_i$.

2.4. **Combinatorics of stable graphs.** To separate the analytic part of perturbation theories, we recall some basic combinatorics of graphs. For more detailed explanations, please refer [Cos11].

Definition 2.2. A stable graph is a non-directed graph γ, possibly with external edges; and for each vertex v of γ an element $g(v) \in \mathbb{Z}_{\geqslant 0}$, called the genus of the vertex v; with the property that every vertex of genus 0 is at least trivalent, and every vertex of genus 1 is at least 1-valent.

If γ is a stable graph, the genus $g(\gamma)$ of γ is defined by

$$g(\gamma) = b_1(\gamma) + \sum_{v \in V(\gamma)} g(v)$$

where $b_1(\gamma)$ is the first Betti number of γ, $V(\gamma)$ is the set of all vertices of γ. An ordering of γ is an ordering of the external edges, internal edges and vertices of γ. We use $\mathrm{Ord}(\gamma)$ to denote the set of all ordering of γ.

Definition 2.3. An automorphism of a stable graph γ is an automorphism of graph which preserve the genus of vertices of γ. We denote the set of automorphisms by $\mathrm{Aut}(\gamma)$.

Let

$$\mathcal{O}^+(\mathcal{E})[[\hbar]] \subset \mathcal{O}(\mathcal{E})[[\hbar]]$$

be the subspace of those functionals which are at least cubic modulo $\hbar$.

If $I \in \mathcal{O}(\mathcal{E})[[\hbar]]$, we have

$$I = \sum_{i,k \geqslant 0} \hbar^i I_{i,k},$$

where $I_{i,k}$ is homogeneous of degree k as a polynomial on $\mathcal{E}$.

Let γ be a stable graph, with k external edges. We choose an ordering of γ. We will define

$$w_\gamma(P_{\epsilon<L}, I) \in \mathrm{Sym}^k(\mathcal{E}^\vee) \subset \mathrm{Hom}(\mathcal{E}^{\hat{\otimes} k}, \mathbf{C}).$$

The rule is as follows. Let $H(\gamma)$, $T(\gamma)$, $E(\gamma)$, and $V(\gamma)$ refer to the sets of half-edges, external edges, internal edges, vertices of γ, respectively. Putting a propagator $P_{\epsilon<L}$ at each internal edge of γ, putting a_i at ith external edge of γ, we can obtain an element of

$$\mathcal{E}^{\hat{\otimes}|E(\gamma)|} \otimes \mathcal{E}^{\hat{\otimes}|E(\gamma)|} \otimes \mathcal{E}^{\hat{\otimes}|T(\gamma)|} \cong \mathcal{E}^{\hat{\otimes}|H(\gamma)|}.$$

Putting $I_{i,k}$ at each vertex of valency k and genus i gives us an element of

$$\mathrm{Hom}(\mathcal{E}^{\hat{\otimes}|H(\gamma)|}, \mathbf{C}).$$

Contracting these two elements yields a number $\tilde{w}_\gamma(P_{\epsilon<L}, I)(a_1, \ldots, a_k)$. This define an element

$$\tilde{w}_\gamma(P_{\epsilon<L}, I) \in \mathrm{Hom}(\mathcal{E}^{\hat{\otimes} k}, \mathbf{C}).$$

We define

$$w_\gamma(P_{\epsilon<L}, I) \in \mathrm{Sym}^k(\mathcal{E}^\vee)$$

as the symmetrization of $\tilde{w}_\gamma(P_{\epsilon<L}, I)$.

Definition 2.4. The Feynman graph expansion $W(-,-)$ is defined by the following formula:

$$W(P_{\epsilon<L}, I) = \sum_\gamma \frac{1}{|\operatorname{Aut}(\gamma)|} \hbar^{g(\gamma)} w_\gamma(P_{\epsilon<L}, I) \in \mathcal{O}^+(\mathcal{E})[[\hbar]].$$

To write down the quantum master equation in terms of Feynman graph expansion, we introduce the following technical definitions:

Definition 2.5. Let γ be a stable graph, $e \in E(\gamma)$, $v \in V(\gamma)$, $P' \in \mathrm{Sym}^2(\mathcal{E})$, and $I' \in \mathcal{O}^+(\mathcal{E})[[\hbar]]$.

1. We define $w_{\gamma,e}(P_{\epsilon<L}, P', I)$ in the same way as $w_\gamma(P_{\epsilon<L}, I)$, except the distinguished edge e is labeled by P', whereas all other edges are labelled by P.
2. We define $w_{\gamma,v}(P_{\epsilon<L}, I, I')$ in the same way as $w_\gamma(P_{\epsilon<L}, I)$, except the distinguished vertex v is labeled by I', whereas all other vertices are labelled by P.
3. We use $\operatorname{Aut}(\gamma, e)$ to denote the set of automorphisms which preserve e. Likewise, we use $\operatorname{Aut}(\gamma, v)$ to denote the set of automorphisms which preserve v.

We have the following lemma:

Lemma 2.6. *Let γ be a stable graph, We define $w_{\gamma,e}(P_{\epsilon<L}, Q(P_{\epsilon<L}), I)$ The following are true:*

1.

$$\begin{aligned} & QW(P_{\epsilon<L}, I) \\ = & \sum_{\gamma,e} \frac{1}{|\operatorname{Aut}(\gamma,e)|} \hbar^{g(\gamma)} w_{\gamma,e}(P_{\epsilon<L}, Q(P_{\epsilon<L}), I) \\ + & \sum_{\gamma,v} \frac{1}{|\operatorname{Aut}(\gamma,v)|} \hbar^{g(\gamma)} w_{\gamma,v}(P_{\epsilon<L}, I, QI). \end{aligned}$$

2.

$$\begin{aligned} & \frac{1}{2}\{W(P_{\epsilon<L}, I), W(P_{\epsilon<L}, I)\}_L + \hbar \triangle_L W(P_{\epsilon<L}, I) \\ = & \sum_{\gamma,e} \frac{1}{|\operatorname{Aut}(\gamma,e)|} \hbar^{g(\gamma)} w_{\gamma,e}(P_{\epsilon<L}, K_L, I). \end{aligned}$$

3.

$$\begin{aligned}&\sum_{\gamma,v}\frac{1}{|\operatorname{Aut}(\gamma,v)|}\hbar^{g(\gamma)}w_{\gamma,v}\left(P_{\epsilon<L},I,\frac{1}{2}\{I,I\}\right)\\ =\ &\lim_{t\to 0}\sum_{\gamma,e}\frac{1}{|\operatorname{Aut}(\gamma,e)|}\hbar^{g(\gamma)}w_{\gamma,e}(P_{\epsilon<L},K_t,I)\end{aligned}$$

Proof. The first two identities can be proven by graph combinatorics. The first two identities can be proven by graph combinatorics. As an example, We prove the first identity. For a stable graph γ, we have

$$Qw_\gamma(P_{\epsilon<L},I)=\sum_{e\in E(\gamma)}w_{\gamma,e}(P_{\epsilon<L},Q(P_{\epsilon<L}),I)+\sum_{v\in V(\gamma)}w_{\gamma,v}(P_{\epsilon<L},I,QI)$$

There is an action of the automorphism group $\operatorname{Aut}(\gamma)$ on $E(\gamma)$. The quotient set is the isomorphism class of stable graphs with a special edge such that the underling graph is γ. We use $E'(\gamma)$ to denote this set. Likewise, we use $V'(\gamma)$ to denote quotient set of $\operatorname{Aut}(\gamma)$ on $V(\gamma)$. Then we have

$$\begin{aligned}&Qw_\gamma(P_{\epsilon<L},I)\\ =\ &\sum_{e\in E'(\gamma)}\frac{|\operatorname{Aut}(\gamma)|}{|\operatorname{Aut}(\gamma,e)|}w_{\gamma,e}(P_{\epsilon<L},Q(P_{\epsilon<L}),I)+\sum_{v\in V'(\gamma)}\frac{|\operatorname{Aut}(\gamma)|}{|\operatorname{Aut}(\gamma,v)|}w_{\gamma,v}(P_{\epsilon<L},I,QI).\end{aligned}$$

We multiply both sides of above identity by $\frac{1}{|\operatorname{Aut}(\gamma)|}\hbar^{g(\gamma)}$, and sum them over all stable graphs. The final result is the first identity we want to prove.

To prove the last identity, we notice

$$\lim_{t-0}\triangle_t I=0,$$

so the summation of right hand side is nonzero only if e is not a self-loop of γ. Then we notice that there is a bijection f between graphs with one special edge and graphs with one special vertice:

$$f:(\gamma,e)\to(\gamma/e,v_e),$$

where γ/e is the graph obtained by contract e to a single vertex v_e. The last identity follows. □

Corollary 2.7. *Assume we have the classical master equation*

$$QI+\frac{1}{2}\{I,I\}=0.$$

The quantum master equation is satisfied if, for any stable graph γ, the following equality holds:

$$\sum_{e\in E(\gamma)}\left(w_{\gamma,e}(P_{0<L},Q(P_{0<L}),I)+\lim_{t\to 0}\sum_{e\in E(\gamma)}w_{\gamma,e}(P_{0<L},K_t,I)+w_{\gamma,e}(P_{0<L},K_L,I)\right)=0.$$

Proof. Note $P_{0<L}$ is not a smooth function, the existence of

$$w_\gamma(P_{\epsilon<L},I)$$

and

$$w_{\gamma,e}(P_{0<L},Q(P_{0<L}),I)$$

are nontrivial. This follows from our ultraviolet finiteness theorem in later sections. Now, we will assume this fact.

If γ is a stable graph, there is an action of the automorphism group $\mathrm{Aut}(\gamma)$ on $E(\gamma)$. The quotient set is the isomorphism class of stable graphs with a special edge such that the underling graph is γ. We use $E'(\gamma)$ to denote this set. Then we have

$$\begin{aligned}
0 &= \sum_{e\in E(\gamma)}\left(w_{\gamma,e}(P_{0<L},Q(P_{0<L}),I)+\lim_{t\to 0}\sum_{e\in E(\gamma)}w_{\gamma,e}(P_{0<L},K_t,I)+w_{\gamma,e}(P_{0<L},K_L,I)\right)\\
&= \sum_{e\in E'(\gamma)}\frac{|\mathrm{Aut}(\gamma)|}{|\mathrm{Aut}(\gamma,e)|}\Bigg(w_{\gamma,e}(P_{0<L},Q(P_{0<L}),I)+\lim_{t\to 0}\sum_{e\in E(\gamma)}w_{\gamma,e}(P_{0<L},K_t,I)\\
&+ w_{\gamma,e}(P_{0<L},K_L,I)\Bigg)
\end{aligned}$$

If we multiply both sides of above identity by $\frac{1}{|\mathrm{Aut}(\gamma)|}\hbar^{g(\gamma)}$, and sum them over all graphs, we have

$$\begin{aligned}
0 &= \sum_{\gamma,e}\frac{1}{|\mathrm{Aut}(\gamma,e)|}\Bigg(w_{\gamma,e}(P_{0<L},Q(P_{0<L}),I)+\lim_{t\to 0}\sum_{e\in E(\gamma)}w_{\gamma,e}(P_{0<L},K_t,I)\\
&+ w_{\gamma,e}(P_{0<L},K_L,I)\Bigg).
\end{aligned}$$

The quantum master equation follows from Lemma 2.6:

$$QW(P_{0<L},I)+\frac{1}{2}\{W(P_{0<L},I),W(P_{0<L},I)\}_L+\hbar\triangle_L W(P_{0<L},I)=0.$$

□

For the convenience of our proof of main theorems, we notice the following fact:

$$K_t(z-w,x-y)=H(t,z-w,x-y)\otimes\mathfrak{c}_V$$

is a simple tensor in

$$\mathrm{Sym}^2(\mathcal{E}) \cong \mathrm{Sym}^2(\Omega^\bullet(\mathbf{C}^d \times \mathbf{R}^{d'})) \otimes \mathrm{Sym}^2(V),$$

$w_\gamma(P_{\epsilon<L}, I)$ is also a simple tensor:

$$w_\gamma(P_{\epsilon<L}, I) = w_\gamma^{\mathrm{an}}(P_{\epsilon<L}, I) \otimes w_\gamma^{\mathrm{al}}(P_{\epsilon<L}, I).$$

We call $w_\gamma^{\mathrm{an}}(P_{\epsilon<L}, I)$ the analytic part of Feynman graph integral, $w_\gamma^{\mathrm{al}}(P_{\epsilon<L}, I)$ the algrbraic part of Feynman graph integral.

In the following sections, we will only care about the analytic part of Feynman graph integrals.

Remark 2.8. Note I is a local functional which depends only on holomorphic derivatives. The integrand of $w_\gamma^{\mathrm{an}}(P_{\epsilon<L}, I)(a_1, \ldots a_k)$ will be a product of holomorphic derivatives of $P_{\epsilon<L}$ and a smooth differential form with compact support. This motivate the definition of Feynman graph integrals in next section.

3. FEYNMAN GRAPH INTEGALS ON $\mathbf{R}^{d'} \times \mathbf{C}^d$

In this section we study the the analytic part of Feynman graph integrals on $\mathbf{R}^{d'} \times \mathbf{C}^d$. First, we will prove that topological-holomorphic theories are 'UV' finite, in the sense defined below. Then, we obtain some vanishing results for integrals over boundaries of compactified Schwinger spaces which will allow us to prove that topological-holomorphic theories on $\mathbf{R}^{d'} \times \mathbf{C}^d$ admit a quantization to all orders in perturbation theory provided $d' > 1$. We refer to appendix B for a introduction to Schwinger spaces and the sort of compactifications that we make use of in this section. Throughout this section, we will omit the bundle factors (or algebraic factors) that appear in Feynman graph integrals without further mention.

We will use coordinates $z = (z_i)_{1\leqslant i\leqslant d}$ and $x = (x_i)_{1\leqslant j\leqslant d'}$ for holomorphic and smooth coordinates on $\mathbf{C}^d$ and $\mathbf{R}^{d'}$, respectively. The notation

$$\mathrm{d}^d\bar{z}\,\mathrm{d}^{d'}x = \left(\prod_{i=1}^{d}\mathrm{d}\bar{z}_i\right)\left(\prod_{j=1}^{d'}\mathrm{d}x_i\right). \tag{28}$$

refers to the top anti-holomorphic form on $\mathbf{C}^d \times \mathbf{R}^{d'}$. The Dolbeault differential on $\mathbf{C}^d$ is denoted by $\bar{\partial}$ and the de Rham differential on $\mathbf{R}^{d'}$ is denoted simply by d.

Note that any (distributional) differential form on $\mathbf{R}^{d'} \times \mathbf{C}^d$ can be written as

$$\alpha = \sum f_{I\bar{J}K}(z, \bar{z}, x)\mathrm{d}^I z \mathrm{d}^{\bar{J}}\mathrm{d}^K x \tag{29}$$

By an ***anti-holomorphic*** form on $\mathbf{R}^{d'} \times \mathbf{C}^d$ we mean such a form with $I = 0$. We will refer to the ***Hodge type*** of a (distributional) anti-holomorphic differential form as $(|\overline{J}|, |K|)$.

3.1. **Propagators and Feynman graph integrals.** The basic ingredient in the definition of Feynman graph integrals is the propagator, which is a particular distributional valued differential form which is a Green's function for the operator $\overline{\partial} + \mathrm{d}$.

For $p \in \mathbf{R}^{d'} \times \mathbf{C}^d$, recall that the δ-function at p is the form degree $2d + d'$ distribution δ_p defined by $\delta_p(f) = f(p)$ for all $f \in C_c^\infty(\mathbf{R}^{d'} \times \mathbf{C}^d)$. The restriction of δ_0 along the difference map

$$(\mathbf{R}^{d'} \times \mathbf{C}^d) \times (\mathbf{R}^{d'} \times \mathbf{C}^d) \to \mathbf{R}^{d'} \times \mathbf{C}^d, \quad (z, x; w, y) \mapsto (z - w, x - y) \tag{30}$$

is denoted $\delta(z - w, x - y)\mathrm{d}^d(z - w)$.[3] Note that we have extracted the holomorphic d-form component so that $\delta(z - w, x - y)$ has form degree $d + d'$.

Definition 3.1. An ***ordinary propagator*** on $\mathbf{R}^{d'} \times \mathbf{C}^d$ is a distributional valued differential form

$$\widetilde{P}(z - w, \overline{z} - \overline{w}, x - y) \in \mathcal{D}^\bullet(\mathbf{R}^{d'} \times \mathbf{C}^d \times \mathbf{R}^{d'} \times \mathbf{C}^d),$$

such that the following equation holds:

$$(\overline{\partial} + \mathrm{d})\, \widetilde{P}(z - w, \overline{z} - \overline{w}, x - y) = \delta(z - w, \overline{z} - \overline{w}, x - y).$$

Remark 3.2. The form degree of $\widetilde{P}$ is $d + d' - 1$. By construction, $\widetilde{P}$ is an anti-holomorphic distributional form. In the model we will use $\widetilde{P}$ is a sum of forms of Hodge type $(d, d' - 1)$ and $(d - 1, d')$.

The choice of propagator is not unique, but we have the following standard choice.

Definition 3.3. The ***generalized Bochner-Martinelli kernel*** is the distributional form

$$\frac{2^d \Gamma\left(d + \frac{d'}{2}\right)}{\pi^{d + \frac{d'}{2}}} \cdot \frac{1}{(2|z - w|^2 + |x - y|^2)^{d + \frac{d'}{2}}} \cdot$$
$$\left(\sum_{i=1}^{d} (-1)^{i-1} (\overline{z_i - w_i}) \left(\prod_{j \neq i} d(\overline{z_j - w_j}) \right) \mathrm{d}^{d'}(x - y) + \right.$$
$$\left. \sum_{i=1}^{d'} (-1)^{d+i-1} (x_i - y_i) \mathrm{d}^d(\overline{z - w}) \left(\prod_{j \neq i} \mathrm{d}(x_j - y_j) \right) \right)$$

Standard computations give the following.

[3] Sometimes to emphasize the anti-holomorphic dependence we will write $\delta(z - w, \overline{z} - \overline{w}, x - y)$.

Lemma 3.4. *The generalized Bochner–Martinelli kernel is an ordinary propagator on* $\mathbf{R}^{d'} \times \mathbf{C}^d$*. We will denote it by* $\widetilde{P}_{0,+\infty}(z-w, \bar{z}-\overline{w}, x-y)$*.*

This generalized Bochner-Martinelli kernel can be see from the solution to the heat equation. We recall some notations from section 2.3.

Definition 3.5. The ***regularized ordinary propagator*** $\widetilde{P}_{\varepsilon,L}$ is defined by the following formula:

$$\widetilde{P}_{\varepsilon,L}(z-w, \bar{z}-\overline{w}, x-y) = \int_{t=\varepsilon}^{L} (\bar{\partial}_z^* + d_x^*) H(t, z-w, \bar{z}-\overline{w}, x-y)\, \mathrm{d}t \tag{31}$$

defined for $\varepsilon, L > 0$. Here $\bar{\partial}_z^*$ and d_x^* are differential operators acting on the variables z and x, respectively.

Proposition 3.6. *The following equality holds:*

$$\widetilde{P}_{0,+\infty}(z-w, \bar{z}-\overline{w}, x-y) = \lim_{\substack{\varepsilon \to 0 \\ L \to +\infty}} \widetilde{P}_{\varepsilon,L}(z-w, \bar{z}-\overline{w}, x-y). \tag{32}$$

Proof. This can be shown by direct computation. □

We can now define the sort of Feynman graph integrals associated with this choice of propagator. For v an integer we denote by $(z^1, z^2, \ldots, z^v; x^1, \ldots, x^v)$ coordinates on the vth product of spacetime $(\mathbf{R}^{d'} \times \mathbf{C}^d)^v = \mathbf{C}^{dv} \times \mathbf{R}^{d'v}$. In particular, for each $1 \le i \le v$ the pair $(z^i, x^i) = (z_1^i, \ldots, z_d^i; x_1^i, \ldots, x_{d'}^v)$ is a coordinate on $\mathbf{R}^{d'} \times \mathbf{C}^d$. We refer to appendix A for further conventions with graphs that we use in this section.

Definition 3.7. The ***configuration space*** of a graph Γ is

$$\begin{aligned} \mathrm{Conf}(\Gamma) &= \left\{ \left((z^1, x^1), (z^2, x^2), \ldots, \left(z^{|\Gamma_0|}, x^{|\Gamma_0|} \right) \right) \Big| \ (z^i, x^i) \neq (z^j, x^j) \text{ for } i \neq j \right\} \\ &\subset (\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|} \end{aligned}$$

We interpret (z^i, x^i) as the coordinate at the ith vertex of Γ.

We fix the following data:

- A decorated graph (Γ, n) (as explained in appendix A),
- positive numbers $0 < \varepsilon < L$, and
- a smooth, compactly supported, differential form $\Phi \in \Omega_c^\bullet((\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|})$.

The main object of study is the ***Feynman graph integral***

$$W_0^{+\infty}((\Gamma, n), \Phi) \tag{33}$$

defined by

(34)

$$(-1)^{\frac{d+d'-1}{2}|\Gamma_1|(|\Gamma_1|-1)+|\Gamma_1|} \int_{\mathrm{Conf}(\Gamma)} \prod_{e\in\Gamma_1} \partial^{n(e)}_{z^{h(e)}} \widetilde{P}_{\varepsilon,L}(z^{h(e)} - z^{t(e)}, \bar{z}^{h(e)} - \bar{z}^{t(e)}, x^{h(e)} - x^{t(e)}) \wedge \Phi.$$

This integral may not exist, but a regularized version always does.

Definition 3.8. Let (Γ, n) and Φ be as above. We define the ***regularized Feynman graph integral***

$$W_\varepsilon^L((\Gamma, n), \Phi) \tag{35}$$

on $\mathbf{R}^{d'} \times \mathbf{C}^d$ to be the following integral:

(36)

$$(-1)^{\frac{d+d'-1}{2}|\Gamma_1|(|\Gamma_1|-1)+|\Gamma_1|} \int_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \prod_{e\in\Gamma_1} \partial^{n(e)}_{z^{h(e)}} \widetilde{P}_{\varepsilon,L}(z^{h(e)} - z^{t(e)}, \bar{z}^{h(e)} - \bar{z}^{t(e)}, x^{h(e)} - x^{t(e)}) \wedge \Phi.$$

Here $\partial^{n(e)}_{z^{h(e)}} = \partial^{n_{1,e}}_{z_1^{h(e)}} \partial^{n_{2,e}}_{z_2^{h(e)}} \dots \partial^{n_{i,e}}_{z_i^{h(e)}} \dots \partial^{n_{d,e}}_{z_d^{h(e)}}$ is a holomorphic differential operator with constant coefficients which only involves coordinates at the vertex $h(e)$.

The main result of this section is to show that the following.

Theorem 3.9. *The limit*

$$W_0^L((\Gamma, n), \Phi) = \lim_{\varepsilon\to 0} W_\varepsilon^L((\Gamma, n), \Phi), \tag{37}$$

exists.

In other words, we will prove the ultraviolet finiteness of such Feynman graph integrals on $\mathbf{R}^{d'} \times \mathbf{C}^d$. Once we know that this limit exists it follows that the original Feynman graph integral satisfies $W_0^{+\infty}((\Gamma, n), \Phi) < +\infty$. We will prove the existence of this limit by using compactification of Schwinger spaces (see appendix B).

To do this, we recast $W_0^L((\Gamma, n), \Phi)$ in terms of the following propagator in Schwinger spaces:

Definition 3.10. Given $t > 0$, The ***propagator in Schwinger space*** P_t is defined by the following formula:

(38)
$$P_t(z-w, \bar{z}-\overline{w}, x-y) = -\mathrm{d}t \wedge \bar{\partial}_z^* K(t, z-w, \bar{z}-\overline{w}, x-y) + K(t, z-w, \bar{z}-\overline{w}, x-y),$$

or, compactly $P_t = -\mathrm{d}t \wedge \overline{\partial}^* H + H$. We will simply call P_t the propagator if there is no ambiguity.

One important reason to introduce this propagator is the following lemma.

Lemma 3.11. *Let* $u = \frac{\bar{z}-\overline{w}}{2t}$, $v = \frac{x-y}{2\sqrt{t}}$. *Then*

$$P_t(z-w, \bar{z}-\overline{w}, x-y) = \frac{1}{\pi^{d+\frac{d'}{2}}} e^{-(z-w)\cdot u - v\cdot v} \mathrm{d}^d u \mathrm{d}^{d'} v$$

where $(z-w)\cdot u = \sum_{i=1}^d (z_i - w_i) u_i$ *and* $v \cdot v = \sum_{j=1}^{d'} v_j v_j$.

Proof. This can be shown by direct computation. □

We have two additional useful properties for propagator in Schwinger space:

Lemma 3.12.

1. *Let* d_t *be the de Rham differential on Schwinger space, then*

(39)
$$(\mathrm{d}_t + \bar{\partial} + d) P_t(z-w, \bar{z}-\overline{w}, x-y) = 0.$$

2. *Define the vector fields* $Eu_t = t\frac{\partial}{\partial t}$, $Eu_{\bar{z}} = \sum_{i=1}^d \bar{z}_i \frac{\partial}{\partial \bar{z}}$, $Eu_x = \sum_{j=1}^{d'} x_j \frac{\partial}{\partial x_j}$. *Then one has the equality*

(40)
$$\left(i_{Eu_t} + i_{Eu_{\bar{z}}} + i_{Eu_{\overline{w}}} + i_{Eu_x} + i_{Eu_y}\right) P_t(z-w, \bar{z}-\overline{w}, x-y) = 0$$

Proof. These can be shown by direct computation. □

With this propagator, we can rephrase the regularized Feynman graph integral $W_\varepsilon^L((\Gamma, n), \Phi)$ in the following way.

Proposition 3.13. *Given decorated graph* (Γ, n) *and* $\Phi \in \Omega_c^\bullet(\mathbf{R}^{d'} \times \mathbf{C}^d)$, *we have the following equality:*

$$W_\varepsilon^L((\Gamma, n), \Phi) = \int_{(\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|} \times [\varepsilon, L]^{|\Gamma_1|}} \prod_{e \in \Gamma_1} \partial_{z^{h(e)}}^{n(e)} P_{t_e}(z^{h(e)} - z^{t(e)}, \bar{z}^{h(e)} - \bar{z}^{t(e)}, x^{h(e)} - x^{t(e)}) \wedge \Phi,$$

where t_e *is the parameter associated with each edge* $e \in \Gamma_1$.

Proof. In the integrand

$$\prod_{e\in\Gamma_1} \partial^{n(e)}_{z^{h(e)}} P_{t_e}(z^{h(e)}-z^{t(e)},\bar{z}^{h(e)}-\bar{z}^{t(e)},x^{h(e)}-x^{t(e)})\wedge\Phi,$$

P_{t_e} appears $|\Gamma_1|$ times and each provides at most a single $\mathrm{d}t_e$ component. Furthermore, the number of propagators equal to the dimension of $[\varepsilon,L]^{|\Gamma_1|}$, so we have the following equality:

$$\begin{aligned}
& \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}\times[\varepsilon,L]^{|\Gamma_1|}} \prod_{e\in\Gamma_1} \partial^{n(e)}_{z^{h(e)}} P_{t_e}(z^{h(e)}-z^{t(e)},\bar{z}^{h(e)}-\bar{z}^{t(e)},x^{h(e)}-x^{t(e)})\wedge\Phi \\
= \;& (-1)^{|\Gamma_1|} \int\limits_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}\times[\varepsilon,L]^{|\Gamma_1|}} \prod_{e\in\Gamma_1} \partial^{n(e)}_{z^{h(e)}} P_{t_e}(z^{h(e)}-z^{t(e)},\bar{z}^{h(e)}-\bar{z}^{t(e)},x^{h(e)}-x^{t(e)})\wedge\Phi \\
= \;& (-1)^{\frac{d+d'-1}{2}|\Gamma_1|(|\Gamma_1|-1)+|\Gamma_1|} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \int\limits_{[\varepsilon,L]^{|\Gamma_1|}} \prod_{e\in\Gamma_1}\mathrm{d}t_e \\
& \times \prod_{e\in\Gamma_1} \partial^{n(e)}_{z^{h(e)}} \bar{\partial}^*_{z^{h(e)}} H(z^{h(e)}-z^{t(e)},\bar{z}^{h(e)}-\bar{z}^{t(e)},x^{h(e)}-x^{t(e)})\wedge\Phi \\
= \;& (-1)^{\frac{d+d'-1}{2}|\Gamma_1|(|\Gamma_1|-1)+|\Gamma_1|} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \prod_{e\in\Gamma_1} \partial^{n(e)}_{z^{h(e)}} \widetilde{P}_{\varepsilon,L}(z^{h(e)}-z^{t(e)},\bar{z}^{h(e)}-\bar{z}^{t(e)},x^{h(e)}-x^{t(e)})\wedge\Phi \\
= \;& W^L_\varepsilon((\Gamma,n),\Phi)
\end{aligned}$$

We have used the fact that only the top differential forms contributes the integral over $[\varepsilon,L]^{|\Gamma_1|}$. □

To prove the finiteness property, we need to realize Feynman graph integral as a integral of a differential form over Schwinger spaces:

Proposition 3.14. *Given decorated graph (Γ,n) and $\Phi\in\Omega^\bullet_c((\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|})$, we denote the integrand*

$$\prod_{e\in\Gamma_1} \partial^{n(e)}_{z^{h(e)}} P_{t_e}(z^{h(e)}-z^{t(e)},\bar{z}^{h(e)}-\bar{z}^{t(e)},x^{h(e)}-x^{t(e)})\wedge\Phi$$

by $\widetilde{W}((\Gamma,n),\Phi)$. Then

$$\int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),\Phi)$$

is a smooth differential form on $(0,L]^{|\Gamma_1|}$ for any $L>0$.

Proof. This can be proved by dominated convergence theorem easily. We will prove a stronger version later. See Proposition 3.18. □

Our strategy to prove the finiteness of $W_\varepsilon^L((\Gamma,n),\Phi)$ can be described by the following two steps:

1. Prove $\int_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}}\widetilde{W}((\Gamma,n),\Phi)$ can be extended to a differential form over the compactification $\widetilde{[0,L]^{|\Gamma_1|}}$,
2. Since $\widetilde{[0,L]^{|\Gamma_1|}}$ is compact, the integral over $\widetilde{[0,L]^{|\Gamma_1|}}$ is automatically finite.

Before starting our main analysis, we rephrase Feynman graph integrals using a convenient coordinate system. We take note of the following facts. If a decorated graph (Γ,n) have two connected components $(\Gamma', n|_{\Gamma'})$ and $(\Gamma'', n|_{\Gamma''})$, we have

$$W_\varepsilon^L((\Gamma,n),\Phi'\cdot\Phi'') = \pm W_\varepsilon^L((\Gamma',n|_{\Gamma'}),\Phi')W_\varepsilon^L((\Gamma'',n|_{\Gamma''}),\Phi''),$$

where $\Phi'\in\Omega_c^\bullet((\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0'|})$, $\Phi''\in\Omega_c^{*,*}((\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0''|})$. Furthermore, if a decorated graph (Γ,n) contains a self-loop, then

$$W_\varepsilon^L((\Gamma,n),\Phi)=0.$$

Therefore, without loss of generality, we assume that Γ is connected without self-loops in what follows.

Recall that we denote the coordinate on the ith factor in

$$(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|} = (\mathbf{R}^{d'}\times\mathbf{C}^d)\times\cdots\times(\mathbf{R}^{d'}\times\mathbf{C}^d) \tag{41}$$

by

$$(z^i,x^i) = (z_1^i,\ldots,z_d^i;x_1^i,\ldots,x_{d'}^i). \tag{42}$$

Given a labeled graph (Γ,n), we introduce the following coordinates:

$$\begin{cases} z^i = w^i + w^{|\Gamma_0|}, x^i = q^i + q^{|\Gamma_0|} & 1\leqslant i\leqslant |\Gamma_0|-1 \\ z^{|\Gamma_0|} = w^{|\Gamma_0|}, x^{|\Gamma_0|} = q^{|\Gamma_0|} \end{cases}$$

Here we have used the ordering of vertices of Γ.

Using lemma 3.11 and coordinates $(w_j^i;q_j^i)$ for (41), the integrand of Feynman graph integral becomes:

$$\widetilde{W}((\Gamma,n),\Phi) = \frac{1}{\pi^{\left(d+\frac{d'}{2}\right)|\Gamma_1|}} e^{-\sum\limits_{i=1}^{|\Gamma_0|-1}\sum\limits_{e\in\Gamma_1}\rho_i^e w^i\cdot u^e - \sum\limits_{e\in\Gamma_1} v^e\cdot v^e} \prod_{e\in\Gamma_1}\left(\prod_{1\leqslant i\leqslant d}(u_i^e)^{n_{i,e}}\right)\mathrm{d}^d u^e \mathrm{d}^{d'} v^e\wedge\Phi, \tag{43}$$

where

$$u^e = \sum_{i=1}^{|\Gamma_0|-1}\frac{1}{2t_e}\rho_i^e\overline{w}^i,\qquad v^e = \sum_{i=1}^{|\Gamma_0|-1}\frac{1}{2\sqrt{t_e}}\rho_i^e q^i. \tag{44}$$

The proof of the following proposition shows the utility of this change of coordinates.

Proposition 3.15. *If there exists a connected subgraph* $\Gamma' \subseteq \Gamma$*, such that*

$$(d+d')|\Gamma'_0| < (d+d'-1)|\Gamma'_1| + d + d' + 1,$$

then $\widetilde{W}((\Gamma, n), \Phi) = 0$*.*

Proof. Note there is a factor $\widetilde{W}((\Gamma', n|_{\Gamma'}), 1)$ in $\widetilde{W}((\Gamma, n), \Phi)$, we only need to prove $\widetilde{W}((\Gamma', n|_{\Gamma'}), 1) = 0$.

Consider the map

$$g_{\Gamma'} \colon (\mathbf{R}^{d'} \times \mathbf{C}^{d})^{|\Gamma'_0|} \times (0, +\infty)^{|\Gamma'_1|} \to (\mathbf{C}^{d} \times \mathbf{R}^{d'})^{|\Gamma'_1|} \tag{45}$$

defined by

$$g_{\Gamma'}(w^i, \overline{w}^i, q^i, t_e) = (u^e, v^e).$$

Since $g_{\Gamma'}$ is a anti-holomorphic map with respect to variables $(w^i, \overline{w}^i)_{i \in \Gamma'_0}$, its anti-holomorphic derivative is of the form:

$$\overline{D}_{(w^i, \overline{w}^i, q^i, t_e)} g \colon T^{0,1}_{(w^i, \overline{w}^i, q^i)} (\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma'_0|} \oplus T_{t_e}(0, +\infty)^{|\Gamma'_1|} \to T^{1,0}_{(u^e, v^e)} (\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma'_1|}.$$

Here $T^{0,1}_{\bullet}$ is the tangent space spanned by anti-holomorphic derivatives in $\overline{w}$ and derivatives in x, and $T^{1,0}_{\bullet}$ is spanned by holomorphic derivatives in u^e and v^e.

The following vectors belong to the kernel of $\overline{D}_{(w^i, \overline{w}^i, q^i, t_e)} g$:

$$\sum_{i=1}^{|\Gamma'_0|} \partial_{w^i_j}, \quad \sum_{i=1}^{|\Gamma'_0|} \partial_{x^i_{j'}}, \quad \sum_{i=1}^{|\Gamma_1|} t_e \partial_{t_e} + \sum_{i,k} \overline{w}^i_k \partial_{\overline{w}^i_k} + \frac{1}{2} \sum_{\substack{1 \leqslant i \leqslant |\Gamma_0| \\ 1 \leqslant k \leqslant d'}} x^i_k \partial_{x^i_k},$$

where $1 \leqslant j \leqslant d,\ 1 \leqslant j' \leqslant d'$. So the rank of this map is bounded above by

$$(d+d')|\Gamma'_0| + |\Gamma'_1| - d - d' - 1 < (d+d')|\Gamma'_1|.$$

On the other hand, the dimension of $T_{(u^e, v^e)}(\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma'_1|}$ is $(d+d')|\Gamma'_1|$. We conclude that $\overline{D}_{(w^i, \overline{w}^i, q^i, t_e)} g$ is not surjective.

Now, notice that $\widetilde{W}((\Gamma', n|_{\Gamma'}), 1)$ contains a factor $g^*_{\Gamma'}\left(\prod_{e \in \Gamma'_1} \mathrm{d}^d u^e \mathrm{d}^{d'} v^e\right)$. Since $\prod_{e \in \Gamma'_1} \mathrm{d}^d u^e \mathrm{d}^{d'} v^e$ is a top holomorphic form on $(\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma'_1|}$,

$$g^*_{\Gamma'}\left(\prod_{e \in \Gamma'_1} \mathrm{d}^d u^e \mathrm{d}^{d'} v^e\right)_{(w^i, \overline{w}^i, x^i, t_e)} = 0$$

since $\overline{D}_{(w^i, \overline{w}^i, q^i, t_e)} g$ is not surjective. We conclude that $\widetilde{W}((\Gamma, n), \Phi) = 0$. □

Following (43), we can rephrase the Feynman graph integral:

$$W_\varepsilon^L((\Gamma,n),\Phi) = \frac{(-1)^{d'^2(|\Gamma_0|-1)}}{\pi^{\left(d+\frac{d'}{2}\right)|\Gamma_1|}} \int\limits_{\left(w^{|\Gamma_0|},q^{|\Gamma_0|}\right)} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|-1}\times[\varepsilon,L]^{|\Gamma_1|}} e^{-\sum\limits_{i=1}^{|\Gamma_0|-1}\sum\limits_{e\in\Gamma_1}\rho_i^e w^i\cdot u^e - \sum\limits_{e\in\Gamma_1} v^e\cdot v^e} \prod_{e\in\Gamma_1}\left(\prod_{1\leqslant i\leqslant d}(u_i^e)^{n_{i,e}}\right) \mathrm{d}^d u^e \mathrm{d}^{d'} v^e \wedge \Phi, \tag{46}$$

where $\int_{\left(w^{|\Gamma_0|},q^{|\Gamma_0|}\right)}$ denotes integration over the $|\Gamma_0|$-vertex

$$\left(w^{|\Gamma_0|},q^{|\Gamma_0|}\right) \in \mathbf{R}^{d'}\times\mathbf{C}^d.$$

Notice that the integrand in (46) is a smooth differential form in the variables (w^i,x^i,u^e,v^e). Our goal is to utilize a compactification of Schwinger space such that these coordinate functions extend smoothly. This will allow us to extend

$$\int\limits_{\left(w^{|\Gamma_0|},q^{|\Gamma_0|}\right)} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|-1}} \widetilde{W}((\Gamma,n),\Phi)$$

to a smooth differential form on the compactification. This will be achieved in next subsection.

3.2. **Finiteness of Feynman graph integrals.** In this subsection, we will prove the main theorem on finiteness of Feynman graph integrals on $\mathbf{R}^{d'}\times\mathbf{C}^d$. The key idea is find a "coordinate transformation" of $(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}\times(0,+\infty)^{|\Gamma_1|}$, such that the integrand $\widetilde{W}((\Gamma,n),\Phi)$ can be expressed in terms of $(M_\Gamma(t)^{-1})^i_j$, $(d_\Gamma(t)^{-1})^{ej}$ and their de Rham differentials. We refer to appendix A for the definitions of $M_\Gamma(t)$ and $\mathrm{d}_\Gamma(t)$.

We start from purely topological case where the underlying space is $\mathbf{R}^{d'}$, so $d=0$ and consider the new coordinates $(\widetilde{q}^i,\widetilde{t}_e)$ defined by

$$q^i = \sum_{j=1}^{|\Gamma_0|-1}(M_\Gamma(\widetilde{t})^{-1})^i_j\widetilde{q}^j\ ,\quad t_e=\widetilde{t}_e^2 \tag{47}$$

where $1\leqslant i\leqslant|\Gamma_0|-1$ and $e\in\Gamma_1$. This change of coordinates covers the map (see lemma B.10 for a definition)

$$t_{\text{square}}\Big|_{(0,+\infty)^{|\Gamma_1|}} : (0,+\infty)^{|\Gamma_1|}\to(0,+\infty)^{|\Gamma_1|}.$$

In these new coordinates we have the following expression for v^e, see (44):

$$
\begin{aligned}
v^e &= \sum_{i=1}^{|\Gamma_0|-1} \frac{1}{2\sqrt{t_e}} \rho^e_i q^i \\
&= \sum_{i=1}^{|\Gamma_0|-1} \sum_{j=1}^{|\Gamma_0|-1} \frac{1}{2\widetilde{t}_e} \rho^e_i (M_\Gamma(\widetilde{t})^{-1})^i_j \widetilde{q}^j \\
&= \sum_{j=1}^{|\Gamma_0|-1} \frac{1}{2} (d_\Gamma(\widetilde{t})^{-1})^{ej} \widetilde{q}^j. \qquad (48)
\end{aligned}
$$

By lemma B.9, the integrand

$$
\widetilde{W}_{\mathrm{top}}((\Gamma, n), \Phi_{\mathrm{top}}) = \frac{1}{\pi^{\left(\frac{d'}{2}\right)|\Gamma_1|}} e^{-\sum\limits_{e\in\Gamma_1} v^e \cdot v^e} \prod_{e\in\Gamma_1} \mathrm{d}^{d'} v^e \wedge \Phi\left(\sum_{j=1}^{|\Gamma_0|-1} (M_\Gamma(\widetilde{t})^{-1})^i_j \widetilde{q}^j \ , \ q^{|\Gamma_0|} \right)
$$

can be extended to $(\mathbf{R}^{d'})^{|\Gamma_0|} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}$, where $\Phi_{\mathrm{top}} \in \Omega^\bullet_c((\mathbf{R}^{d'})^{|\Gamma_0|})$. As $(\mathbf{R}^{d'})^{|\Gamma_0|-1}$ is non-compact, we will need to use dominated convergence theorem to prove that the differential form

$$
\left(t_{\mathrm{square}} \Big|_{(0,+\infty)^{|\Gamma_1|}} \right)^* \int\limits_{(\mathbf{R}^{d'})^{|\Gamma_0|-1}} \widetilde{W}_{\mathrm{top}}((\Gamma, n), \Phi_{\mathrm{top}}) \qquad (49)
$$

can be extended to $\mathbf{R}^{d'}_{q^{|\Gamma_0|}} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}$. For notational convenience, we will use

$$
t^*_{\mathrm{square}} \int\limits_{(\mathbf{R}^{d'})^{|\Gamma_0|-1}} \widetilde{W}_{\mathrm{top}}((\Gamma, n), \Phi_{\mathrm{top}}) \qquad (50)
$$

in place of the expression (49)

Lemma 3.16. *If we use $M_\Gamma(\widetilde{t})$ to denote the matrix with matrix elements $M_\Gamma(t)^i_j$, we have the following inequality:*

$$
M_\Gamma(\widetilde{t})^{-1} M_\Gamma(\widetilde{t}^2) M_\Gamma(\widetilde{t})^{-1} \geqslant \frac{1}{c_\Gamma} \mathrm{Id},
$$

where c_Γ is a constant which only depends on the graph Γ, and the inequality uses the order structure of symmetric matrices.

Proof. We use ρ to denote the $|\Gamma_1| \times (|\Gamma_0| - 1)$ matrix with entries ρ_i^e, and use $\tilde{t}$ to denote the diagonal $|\Gamma_1| \times |\Gamma_1|$ matrix with diagonal entries $\widetilde{t}_e$. We have

$$\begin{aligned} & M_\Gamma(\widetilde{t})^{-1} M_\Gamma(\widetilde{t^2}) M_\Gamma(\widetilde{t})^{-1} \\ = \; & M_\Gamma(\widetilde{t})^{-1} \rho^T \widetilde{t^2} \rho M_\Gamma(\widetilde{t})^{-1} \\ = \; & M_\Gamma(\widetilde{t})^{-1} \rho^T \widetilde{t}\widetilde{t} \rho M_\Gamma(\widetilde{t})^{-1} \\ \geqslant \; & \frac{1}{c_\Gamma} M_\Gamma(\widetilde{t})^{-1} \rho^T \widetilde{t} \rho \rho^T \widetilde{t} \rho M_\Gamma(\widetilde{t})^{-1} \\ = \; & \frac{1}{c_\Gamma} M_\Gamma(\widetilde{t})^{-1} M_\Gamma(\widetilde{t}) M_\Gamma(\widetilde{t}) M_\Gamma(\widetilde{t})^{-1} \\ = \; & \frac{1}{c_\Gamma} \mathrm{Id}\,. \end{aligned}$$

We have used the following inequality:

$$\rho\rho^T \leqslant C_\Gamma \,\mathrm{Id},$$

where

$$C_\Gamma = |\Gamma_1| \max_{e,e' \in \Gamma_1} \{ |(\rho\rho^T)^{ee'}| \}.$$

□

With this lemma, we can the following result:

Proposition 3.17. *Let (Γ, n) be a connected decorated graph without self-loops, and $\Phi_{\mathrm{top}} \in \Omega_c^\bullet((\mathbf{R}^{d'})^{|\Gamma_0|})$. Then*

$$t_{\mathrm{square}}^* \int_{(\mathbf{R}^{d'})^{|\Gamma_0|-1}} \widetilde{W}_{\mathrm{top}}((\Gamma, n), \Phi_{\mathrm{top}}) \tag{51}$$

can be extended to a smooth form on $\mathbf{R}^{d'}_{q^{|\Gamma_0|}} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}$.

Furthermore, the map

$$\Omega_c^\bullet((\mathbf{R}^{d'})^{|\Gamma_0|}) \to \Omega^\bullet\left(\mathbf{R}^{d'}_{q^{|\Gamma_0|}} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}\right) \tag{52}$$

which sends Φ_{top} to (51) is a continuous map between topological vector spaces.

Proof. By lemma B.9 and (49), we know $\widetilde{W}_{\mathrm{top}}((\Gamma, n), \Phi_{\mathrm{top}})$ can be extended to $(\mathbf{R}^{d'})^{|\Gamma_0|} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}$.

Now, we note $\widetilde{W}_{\text{top}}((\Gamma,n),\Phi_{\text{top}})$ can be rewrited in the following form:

$$\widetilde{W}_{\text{top}}((\Gamma,n),\Phi_{\text{top}}) = e^{-\frac{1}{4}\sum\limits_{i=1}^{|\Gamma_0|-1}\widetilde{q}^i\cdot(M_\Gamma(\widetilde{t})^{-1}M_\Gamma(\widetilde{t}^2)M_\Gamma(\widetilde{t})^{-1})_{ij}\widetilde{q}^j}P(\widetilde{q}^i)\omega\wedge\Phi_{\text{top}}, \tag{53}$$

where ω is a constant differential form and P is a polynomial with coefficients in terms of $(M_\Gamma(t)^{-1})^{ij}$, $(d_\Gamma^{-1})^{ej}$ and their derivatives.

Since $\left[0,\widetilde{\sqrt{L}}\right]^{|\Gamma_1|}$ is compact, the values of $(M_\Gamma(\widetilde{t})^{-1})^i_j$, $(d_\Gamma(\widetilde{t})^{-1})^{ej}$ and their derivatives with respect to $\{\widetilde{t}_e\}$ are bounded by some constant C. So if D is any order $|D|$ differential operator on $(\mathbf{R}^{d'})^{|\Gamma_0|}\times\left[0,\widetilde{\sqrt{L}}\right]^{|\Gamma_1|}$, we have

$$\begin{aligned} D(\widetilde{W}_{\text{top}}((\Gamma,n),\Phi_{\text{top}})) &= D\left(e^{-\frac{1}{4}\sum\limits_{i=1}^{|\Gamma_0|-1}\sum\limits_{j=1}^{|\Gamma_0|-1}\widetilde{q}^i\cdot(M_\Gamma(\widetilde{t})^{-1}M_\Gamma(\widetilde{t}^2)M_\Gamma(\widetilde{t})^{-1})_{ij}\widetilde{q}^j}P(\widetilde{q}^i)\omega\wedge\Phi_{\text{top}}\right) \\ &= e^{-\frac{1}{4}\sum\limits_{i=1}^{|\Gamma_0|-1}\sum\limits_{j=1}^{|\Gamma_0|-1}\widetilde{q}^i\cdot(M_\Gamma(\widetilde{t})^{-1}M_\Gamma(\widetilde{t}^2)M_\Gamma(\widetilde{t})^{-1})_{ij}\widetilde{q}^j}P'(\widetilde{q}^i)\omega\wedge D'\Phi_{\text{top}} \end{aligned}$$

where D' is another differential operator on $(\mathbf{R}^{d'})^{|\Gamma_0|}\times\left[0,\widetilde{\sqrt{L}}\right]^{|\Gamma_1|}$, $P'(\widetilde{q}^i)$ is a polynomial with coefficients in terms of $(M_\Gamma(t)^{-1})^{ij}$, $(d_\Gamma^{-1})^{ej}$ and their derivatives.

Finally, by lemma 3.16, we can get

$$|D(\widetilde{W}_{\text{top}}((\Gamma,n),\Phi_{\text{top}}))| \leqslant C'e^{-\frac{1}{4c_\Gamma}\sum\limits_{i=1}^{|\Gamma_0|-1}\widetilde{q}^i\cdot\widetilde{q}^i}|P'(\widetilde{q}^i)|\max\{|D'\Phi_{\text{top}}|\}, \tag{54}$$

where C' is some constant. Since the right hand side of (54) is absolute integrable and independent of $(\mathbf{R}^{d'})^{|\Gamma_0|}\times\left[0,\widetilde{\sqrt{L}}\right]^{|\Gamma_1|}$, the expression

$$t^*_{\text{square}}\int\limits_{(\mathbf{R}^{d'})^{|\Gamma_0|-1}}\widetilde{W}_{\text{top}}((\Gamma,n),\Phi_{\text{top}}) \tag{55}$$

is smooth over $(\mathbf{R}^{d'})^{|\Gamma_0|}\times\left[0,\widetilde{\sqrt{L}}\right]^{|\Gamma_1|}$ by dominated convergence theorem. □

Now, we can consider Feynman graph integrals on $(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}$.

Proposition 3.18. *Given a connected decorated graph (Γ,n) without self-loops, the following statements are true:*

(1) The map

$$\Omega_c^\bullet((\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|}) \to \Omega^\bullet\left((\mathbf{C}^d \times \mathbf{R}^{d'})_{\left(w^{|\Gamma_0|}, q^{|\Gamma_0|}\right)} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}\right) \tag{56}$$

which sends Φ to the differential form

$$t^*_{\text{square}} \int\limits_{(\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|-1}} \widetilde{W}((\Gamma, n), \Phi) \tag{57}$$

is a continuous map between topological vector spaces.

(2) The map

$$\Omega_c^\bullet((\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|}) \to \Omega^\bullet\left(\widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}\right) \tag{58}$$

which sends Φ to the differential form

$$t^*_{\text{square}} \int\limits_{(\mathbf{C}^d \times \mathbf{R}^{d'})_{\left(w^{|\Gamma_0|}, q^{|\Gamma_0|}\right)}} \int\limits_{(\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|-1}} \widetilde{W}((\Gamma, n), \Phi) \tag{59}$$

is also a continuous map between topological vector spaces.

Proof. (1) Consider first the case $d' = 0$. In this case, it is shown in [Wan24] that the map

$$\Omega_c^\bullet((\mathbf{C}^d)^{|\Gamma_0|}) \to \Omega^\bullet\left(\mathbf{C}^d_{w^{|\Gamma_0|}} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}\right). \tag{60}$$

which sends Φ_{hol} to

$$t^*_{\text{square}} \int\limits_{(\mathbf{C}^d)^{|\Gamma_0|-1}} \widetilde{W}_{hol}((\Gamma, n), \Phi_{hol}) \tag{61}$$

is continuous.

Note the following Künneth isomorphisms between topological vector spaces:

$$\Omega_c^\bullet((\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|}) \cong \Omega_c^\bullet((\mathbf{C}^d)^{|\Gamma_0|}) \hat{\otimes} \Omega_c^\bullet((\mathbf{R}^{d'})^{|\Gamma_0|}),$$

$$\begin{aligned} &\Omega^\bullet\left((\mathbf{R}^{d'} \times \mathbf{C}^d)_{\left(w^{|\Gamma_0|}, q^{|\Gamma_0|}\right)} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}\right) \\ = \; &\Omega^\bullet\left(\mathbf{C}^d_{w^{|\Gamma_0|}} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}\right) \hat{\otimes} \Omega^\bullet\left(\mathbf{R}^{d'}_{q^{|\Gamma_0|}} \times \widetilde{\left[0, \sqrt{L}\right]}^{|\Gamma_1|}\right), \end{aligned}$$

where $\hat{\otimes}$ is the complete projective tensor product.

Combine Proposition 3.17 and the above isomorphisms, we have the continuous map between topological vector spaces

$$\Omega_c^\bullet((\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|})\to\Omega^\bullet\left((\mathbf{C}^d\times\mathbf{R}^{d'})_{(w^{|\Gamma_0|},q^{|\Gamma_0|})}\times\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}\times\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}\right).$$

which sends $\Phi_{hol}\otimes\Phi_{top}$ to

$$t^*_{\text{square}}\int_{(\mathbf{C}^d)^{|\Gamma_0|-1}}\widetilde{W}_{\text{hol}}((\Gamma,n),\Phi_{\text{hol}})\otimes t^*_{\text{square}}\int_{(\mathbf{R}^{d'})^{|\Gamma_0|-1}}\widetilde{W}_{\text{top}}((\Gamma,n),\Phi_{\text{top}}) \tag{62}$$

Composition with restriction along the diagonal yields the desired map.

(2) Note that integration over $(\mathbf{R}^{d'}\times\mathbf{C}^d)_{(w^{|\Gamma_0|},q^{|\Gamma_0|})}$ is a continuous

$$\Omega_c^\bullet\left((\mathbf{C}^d\times\mathbf{R}^{d'})_{(w^{|\Gamma_0|},q^{|\Gamma_0|})}\times\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}\right)\to\Omega^\bullet\left(\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}\right) \tag{63}$$

Since the support of $t^*_{\text{square}}\int_{(\mathbf{R}^{d'})^{|\Gamma_0|-1}}\widetilde{W}_{\text{top}}((\Gamma,n),\Phi_{\text{top}})$ is compact, our claim follows from proposition 3.17.

□

We arrive at the main result and a strengthening of theorem 3.9.

Corollary 3.19. *Given a connected decorated graph (Γ,n) without self-loops, and $\Phi\in\Omega_c^\bullet((\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|})$, we have $W_0^L((\Gamma,n),\Phi)<+\infty$. Furthermore, the map*

$$\Omega_c^\bullet((\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|})\to\mathbf{C}$$

which sends Φ to $W_0^L((\Gamma,n),\Phi)$ defines a distribution on $(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}$.

Proof. Integration over $\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}$ is a continuous map

$$\Omega^\bullet\left(\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}\right)\to\mathbf{C}. \tag{64}$$

The result is then a consequence of proposition 3.18. □

3.3. **Anomaly integrals.** By Corollary 2.7, it is natural to compute

$$(\bar\partial+d)W_0^L((\Gamma,n),-), \tag{65}$$

Which should relate to the quantum master equation. We will identify the failure of the equation

$$(\bar\partial+d)W_0^L((\Gamma,n),-)=0 \tag{66}$$

as integrals over boundaries of compactified Schwinger spaces.

3.3.1. *Integrals over boundaries of compactified Schwinger spaces.*

Proposition 3.20. *Given a connected decorated graph (Γ, n) without self-loops, and $\Phi \in \Omega_c^\bullet((\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|})$, $L > 0$, the following equality holds:*

$$
\begin{aligned}
&((\bar\partial + d) W_0^L)((\Gamma, n), -) \\
= \; &(-1)^{|\Gamma_1|} t_{\text{square}}^* \int_{\partial \widetilde{[0,\sqrt{L}]}^{|\Gamma_1|}} \int_{(\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma, n), -).
\end{aligned}
\tag{67}
$$

Proof. By Lemma 3.12, we have

$$
\begin{aligned}
t_{\text{square}}^* \widetilde{W}((\Gamma, n), (\bar\partial + d)\Phi) \;=\; & (-1)^{(d+d'-1)|\Gamma_1|} (\bar\partial + d) \\
& \left(\prod_{e \in \Gamma_1} \partial_{z^{h(e)}}^{n(e)} P_{\tilde{t}_e^2}(z^{h(e)} - z^{t(e)}, \bar z^{h(e)} - \bar z^{t(e)}, x^{h(e)} - x^{t(e)}) \wedge \Phi \right) \\
- \; & (-1)^{(d+d'-1)|\Gamma_1|} ((\bar\partial + d) \\
& \left(\prod_{e \in \Gamma_1} \partial_{z^{h(e)}}^{n(e)} P_{\tilde{t}_e^2}(z^{h(e)} - z^{t(e)}, \bar z^{h(e)} - \bar z^{t(e)}, x^{h(e)} - x^{t(e)}) \right) \Bigg) \wedge \Phi \\
= \; & (-1)^{(d+d'-1)|\Gamma_1|} (\bar\partial + d) \\
& \left(\prod_{e \in \Gamma_1} \partial_{z^{h(e)}}^{n(e)} P_{\tilde{t}_e^2}(z^{h(e)} - z^{t(e)}, \bar z^{h(e)} - \bar z^{t(e)}, x^{h(e)} - x^{t(e)}) \wedge \Phi \right) \\
+ \; & (-1)^{(d+d'-1)|\Gamma_1|} (d_{\tilde{t}} \\
& \left(\prod_{e \in \Gamma_1} \partial_{z^{h(e)}}^{n(e)} P_{\tilde{t}_e^2}(z^{h(e)} - z^{t(e)}, \bar z^{h(e)} - \bar z^{t(e)}, x^{h(e)} - x^{t(e)}) \right) \Bigg) \wedge \Phi \\
= \; & (-1)^{(d+d'-1)|\Gamma_1|} (\bar\partial + d) \\
& \left(\prod_{e \in \Gamma_1} \partial_{z^{h(e)}}^{n(e)} P_{\tilde{t}_e^2}(z^{h(e)} - z^{t(e)}, \bar z^{h(e)} - \bar z^{t(e)}, x^{h(e)} - x^{t(e)}) \wedge \Phi \right) \\
+ \; & (-1)^{(d+d'-1)|\Gamma_1|} d_{\tilde{t}} (\widetilde{W}((\Gamma, n), \Phi))
\end{aligned}
$$

therefore

$$
\begin{aligned}
((\bar\partial+d)W_0^L)((\Gamma,n),\Phi) &= (-1)^{(d+d')|\Gamma_1|+d'|\Gamma_0|}W_0^L((\Gamma,n),(\bar\partial+d)\Phi)\\
&= (-1)^{(d+d')|\Gamma_1|+d'|\Gamma_0|}\int_{\widetilde{[0,\sqrt{L}]}^{|\Gamma_1|}} t^*_{\mathrm{square}}\int_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}}\widetilde{W}((\Gamma,n),(\bar\partial+d)\Phi)\\
&= (-1)^{|\Gamma_1|+d'|\Gamma_0|}\int_{\widetilde{[0,\sqrt{L}]}^{|\Gamma_1|}} t^*_{\mathrm{square}}\int_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} d_{\tilde t}(\widetilde{W}((\Gamma,n),\Phi))\\
&= (-1)^{|\Gamma_1|}\int_{\widetilde{[0,\sqrt{L}]}^{|\Gamma_1|}} d_{\tilde t}t^*_{\mathrm{square}}\int_{(\mathbf{C}^{d}\times\mathbf{R}^{d'})^{|\Gamma_0|}}\widetilde{W}((\Gamma,n),\Phi)\\
&= (-1)^{|\Gamma_1|}\int_{\partial\widetilde{[0,\sqrt{L}]}^{|\Gamma_1|}} t^*_{\mathrm{square}}\int_{(\mathbf{C}^{d}\times\mathbf{R}^{d'})^{|\Gamma_0|}}\widetilde{W}((\Gamma,n),\Phi).
\end{aligned}
$$

□

We have shown the failure of (66) is given by integrals over the boundary of Schwinger spaces. By Proposition B.11, the boundaries $\partial\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}$ has the following decomposition:

$$
\partial\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|} = \left(-\partial_0\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}\right)\cup\partial_{\sqrt{L}}\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|},
$$

where $\partial_0\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}$($\partial_{\sqrt{L}}\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}$) describe the boundary components near the origin(away from the origin). More precisely, we have

$$
\begin{cases}
\partial_0\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|} = \bigcup_{\Gamma'\subseteq\Gamma}(-1)^{\sigma(\Gamma',\Gamma/\Gamma')}\widetilde{[0,+\infty]}^{|\Gamma_1'|}/\mathbf{R}^+\times\widetilde{\left[0,+\sqrt{L}\right]}^{|\Gamma_1\setminus\Gamma_1'|}\\
\partial_{\sqrt{L}}\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|} = \bigcup_{e\in\Gamma_1}(-1)^{|e|}\left\{\sqrt{L}\right\}\times\widetilde{\left[0,\sqrt{L}\right]}^{|(\Gamma/e)_1|}
\end{cases}.
$$

The integrals over $\partial_{\sqrt{L}}\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}$ reflects global aspects of Feynman graph integrals, and it will not affect the construction of factorization algebras on $\mathbf{R}^{d'}\times\mathbf{C}^d$. The integrals over $\partial_0\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}$ reflects local aspects of Feynman graph integrals, and we need some vanishing results for such integrals to construct factorization algebras.

3.3.2. *Anomaly integrals and Laman graphs.*

In this subsection, we will describe integrals over $\partial_0 \widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|}$ in details and relate them to Laman graphs (see Definition A.10). Given a subgraph $\Gamma' \subseteq \Gamma$, we notice the following equality:

$$\begin{aligned} &\int\limits_{\widetilde{[0,+\infty]}^{|\Gamma_1'|}/\mathbf{R}^+\times\widetilde{[0,+\sqrt{L}]}^{|\Gamma_1\setminus\Gamma_1'|}} t^*_{\text{square}} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),-) \\ = &\int\limits_{\partial C_{\Gamma'}\cap\widetilde{[0,\sqrt{L}]}^{|\Gamma_1|}} t^*_{\text{square}} \int\limits_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),-). \end{aligned}$$

This equality will provide us a way to perform concrete computations. We first concentrate on the case $\Gamma' = \Gamma$. In this case, $\partial C_\Gamma \cap \widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1|} = \partial C_\Gamma$. In the following, we use $O_{(\Gamma,n)}$ to denote the following integral:

$$O_{(\Gamma,n)}(\Phi) = \int\limits_{\partial C_\Gamma} t^*_{\text{square}} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),\Phi), \quad \text{for } \Phi \in \Omega^\bullet_c((\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}).$$

Proposition 3.21. *Given a decorated graph*[4] (Γ,n) *without self-loops, and* Φ *is a compactly supported differential form on* $(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|})$*. The following statements hold:*

1. *If* Γ *is not a Laman graph (see Definition A.10), we have*

$$O_{(\Gamma,n)}(\Phi) = \int\limits_{\partial C_\Gamma} t^*_{\text{square}} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),\Phi) = 0.$$

2. *If* Γ *is a Laman graph,*

$$\begin{aligned} O_{(\Gamma,n)}(\Phi) &= \int\limits_{\partial C_\Gamma} t^*_{\text{square}} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),\Phi) \\ &= \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)_{\left(w^{|\Gamma_0|},q^{|\Gamma_0|}\right)}} (D\Phi)\Big|_{w^i=0,q^i=0,i\neq|\Gamma_0|}, \end{aligned} \tag{68}$$

where D *is a differential operator with constant coefficients, which only involves holomorphic derivatives. The order of* D *is less than or equal to*

$$\sum_{e\in\Gamma_1,1\leqslant i\leqslant d} n_{i,e} + |\Gamma_1| - 1.$$

[4] We do not assume Γ is connected here.

Proof. Let's assume Γ has m connected components $\Gamma^1, \Gamma^2, \ldots, \Gamma^m$. If some Γ^i does not satisfy (71), by Proposition 3.15, we have $\widetilde{W}((\Gamma, n), \Phi) = 0$. Let's assume all Γ^i satisfy (71), then we have

$$(d+d')|\Gamma_0| \geqslant (d+d'-1)|\Gamma_1| + m(d+d'+1).$$

In this case, $M_\Gamma(\widetilde{t}^2)$ is not invertible, but we can easily know that it has exactly $m-1$ zero eigenvectors by studying the weighted Laplacian of each subgraph. We still use $\det\big(M_\Gamma(\widetilde{t}^2)\big)$ to denote the product of its nonzero eigenvalues.

Note $t^*_{\text{square}}\widetilde{W}((\Gamma, n), \Phi)$ has the following form:

$$e^{-\sum\limits_{i=1}^{|\Gamma_0|-1}\sum\limits_{j=1}^{|\Gamma_0|-1}\left(\frac{1}{2}w^i\cdot M_\Gamma(\widetilde{t}^2)_{ij}\overline{w}^j + \frac{1}{4}q^i\cdot M_\Gamma(\widetilde{t}^2)_{ij}q^j\right)} \times$$
$$\left(\sum_{j=0}^{d(|\Gamma_0|-1)} \sum_{\substack{0\leqslant k\leqslant d'(|\Gamma_0|-1),\\ 0\leqslant (d+d')|\Gamma_1|-j-k\leqslant |\Gamma_1|}} P_{j,k}(\overline{w}, x, d\overline{w}, d\vec{x}) \cdot \frac{\widetilde{\widetilde{P}}_{j,k}(d\widetilde{t})}{\widetilde{P}_{j,k}(\widetilde{t})}\right) \wedge \Phi,$$

where $P_{j,k}(\overline{w}, x, d\overline{w}, dx)$ is a homogenous polynomial with variables $\{\overline{w}_i, x, d\overline{w}_i, dx_i\}_{i\in\Gamma_0}$, its degree with respect to $\overline{w}, x, d\overline{w}, dx$ are

$$\sum_{e\in\Gamma_1, 1\leqslant i\leqslant d} n_{i,e} + d|\Gamma_1| - j, \quad d'|\Gamma_1| - k, \quad j, \quad k$$

respectfully. $\widetilde{P}_{j,k}(\widetilde{t})$ is a homogenous polynomial with variables $\{\widetilde{t}_e\}_{e\in\Gamma_1}$, its degree is

$$2\left(\sum_{e\in\Gamma_1, 1\leqslant i\leqslant d} n_{i,e}\right) + (3d+2d')|\Gamma_1| - j - k.$$

$\widetilde{\widetilde{P}}_{j,k}(d\widetilde{t})$ is a homogenous polynomial with variables $\{d\widetilde{t}_e\}_{e\in\Gamma_1}$, its degree is

$$(d+d')|\Gamma_1| - j - k.$$

Using coordinates $\left\{\widetilde{\rho} = \sqrt{\sum\limits_{e\in\Gamma}\widetilde{t}_e^2}, \widetilde{\xi}_e = \frac{\widetilde{t}_e}{\sqrt{\sum\limits_{e\in\Gamma}\widetilde{t}_e^2}}\right\}_{e\in\Gamma_1}$, we can rewrite $t^*_{\text{square}}\widetilde{W}((\Gamma, n), \Phi)$ as

$$e^{-\frac{1}{\widetilde{\rho}^2}\sum\limits_{i=1}^{|\Gamma_0|-1}\sum\limits_{j=1}^{|\Gamma_0|-1}\left(\frac{1}{2}w^i\cdot M_\Gamma(\widetilde{\xi}^2)_{ij}\overline{w}^j + \frac{1}{4}q^i\cdot M_\Gamma(\widetilde{\xi}^2)_{ij}q^j\right)} \times$$
$$\widetilde{\rho}^{-2\left(\sum\limits_{e\in\Gamma_1, 1\leqslant i\leqslant d} n_{i,e}\right) - (3d+2d')|\Gamma_1| + j + k} \times$$
$$\left(\sum_{j=0}^{d(|\Gamma_0|-1)} \sum_{\substack{0\leqslant k\leqslant d'(|\Gamma_0|-1),\\ 0\leqslant (d+d')|\Gamma_1|-j-k\leqslant |\Gamma_1|}} P_{j,k}(\overline{w}, x, d\overline{w}, dx) \cdot \frac{\widetilde{\widetilde{P}}_{j,k}(d(\widetilde{\rho}\widetilde{\xi}))}{\widetilde{P}_{j,k}(\widetilde{\xi})}\right) \wedge \Phi,$$

Note $d\widetilde{\rho}|_{\partial C_\Gamma} = 0$, we have

$$\left(t^*_{\text{square}} \int\limits_{(\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma, n), \Phi)\right)|_{\widetilde{\rho}=0} = \left(\widetilde{\rho}^{\,-2\left(\sum\limits_{e\in\Gamma_1, 1\leqslant i\leqslant d} n_{i,e}\right)-(2d+d')|\Gamma_1|} \times \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} e^{-\frac{1}{\widetilde{\rho}^2}\sum\limits_{i=1}^{|\Gamma_0|-1}\sum\limits_{j=1}^{|\Gamma_0|-1}\left(\frac{1}{2}w^i\cdot M_\Gamma(\widetilde{\xi}^2)_{ij}\overline{w}^j+\frac{1}{4}q^i\cdot M_\Gamma(\widetilde{\xi}^2)_{ij}q^j\right)} \times \left(\sum_{j=0}^{d(|\Gamma_0|-1)} \sum_{\substack{0\leqslant k\leqslant d'(|\Gamma_0|-1),\\ 0\leqslant (d+d')|\Gamma_1|-j-k\leqslant|\Gamma_1|}} P_{j,k}(\overline{w}, x, d\overline{w}, d\vec{x}) \cdot \frac{\widetilde{\widetilde{P}}_{j,k}(d(\widetilde{\xi}))}{\widetilde{P}_{j,k}(\widetilde{\xi})}\right) \wedge \Phi\right)|_{\widetilde{\rho}=0}.$$

We have the following asymptotic formula for Gaussian type integral(see [Dui11, Section 2.3] for a proof) when $\widetilde{\rho} \to 0$:

$$
\widetilde{\rho}^{-2\left(\sum_{e\in\Gamma_1,1\leqslant i\leqslant d} n_{i,e}\right)-(2d+d')|\Gamma_1|} \int_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|-1}} e^{-\frac{1}{\widetilde{\rho}^2}\sum_{i=1}^{|\Gamma_0|-1}\sum_{j=1}^{|\Gamma_0|-1}\left(\frac{1}{2}w^i\cdot M_\Gamma(\widetilde{\xi}^2)_{ij}\overline{w}^j+\frac{1}{4}q^i\cdot M_\Gamma(\widetilde{\xi}^2)_{ij}q^j\right)} \times
$$

$$
\left(\sum_{j=0}^{d(|\Gamma_0|-1)} \sum_{\substack{0\leqslant k\leqslant d'(|\Gamma_0|-1),\\ 0\leqslant(d+d')|\Gamma_1|-j-k\leqslant|\Gamma_1|}} P_{j,k}(\overline{w},\vec{x},d\overline{w},d\vec{x})\cdot\frac{\widetilde{\widetilde{P}}_{j,k}(d(\widetilde{\xi}))}{\widetilde{P}_{j,k}(\widetilde{\xi})}\right)\wedge\Phi
$$

$$
\begin{aligned}
\sim\quad & (2\pi)^{\left(d+\frac{1}{2}d'\right)(|\Gamma_0|-m)}\widetilde{\rho}^{-2\left(\sum_{e\in\Gamma_1,1\leqslant i\leqslant d} n_{i,e}\right)-(2d+d')|\Gamma_1|}\times\widetilde{\rho}^{(2d+d')|\Gamma_0|-m(2d+d')} \\
& \frac{1}{\det\left(M_\Gamma(\widetilde{\xi}^2)\right)^{d+\frac{1}{2}d'}} e^{\widetilde{\rho}^2\sum_{i=1}^{|\Gamma_0|-1}\sum_{j=1}^{|\Gamma_0|-1}\left(\sum_{k=1}^{d}\frac{1}{2}M_\Gamma(\widetilde{\xi}^2)^{-1}_{ij}\partial_{w^i_k}\partial_{\widetilde{w}^j_k}+\sum_{k=1}^{d'}M_\Gamma(\widetilde{\xi}^2)^{-1}_{ij}\partial_{q^i_k}\partial_{q^j_k}\right)} i_{\omega^\vee} \\
& \left(\left(\sum_{j=0}^{d(|\Gamma_0|-1)} \sum_{\substack{0\leqslant k\leqslant d'(|\Gamma_0|-1),\\ 0\leqslant(d+d')|\Gamma_1|-j-k\leqslant|\Gamma_1|}} P_{j,k}(\overline{w},\vec{x},d\overline{w},d\vec{x})\cdot\frac{\widetilde{\widetilde{P}}_{j,k}(d(\widetilde{\xi}))}{\widetilde{P}_{j,k}(\widetilde{\xi})}\right)\wedge\Phi\right) \\
& \Big|_{w^i=0,q^i=0,i\neq|\Gamma_0|} \\
=\quad & (2\pi)^{\left(d+\frac{1}{2}d'\right)(|\Gamma_0|-m)}\widetilde{\rho}^{-2\left(\sum_{e\in\Gamma_1,1\leqslant i\leqslant d} n_{i,e}\right)+(2d+d')(|\Gamma_0|-|\Gamma_1|)-m(2d+d')}\times \\
& \frac{1}{\det\left(M_\Gamma(\widetilde{\xi}^2)\right)^{d+\frac{1}{2}d'}} \sum_{l=\left(\sum_{e\in\Gamma_1,1\leqslant i\leqslant d} n_{i,e}\right)+d|\Gamma_1|-j+\frac{1}{2}(d'|\Gamma_1|-k)}^{+\infty} \\
& \frac{1}{l!}\widetilde{\rho}^{2l}\left(\sum_{i=1}^{|\Gamma_0|-1}\sum_{j=1}^{|\Gamma_0|-1}\left(\sum_{k=1}^{d}\frac{1}{2}M_\Gamma(\widetilde{\xi}^2)^{-1}_{ij}\partial_{w^i_k}\partial_{\widetilde{w}^j_k}+\sum_{k=1}^{d'}M_\Gamma(\widetilde{\xi}^2)^{-1}_{ij}\partial_{q^i_k}\partial_{q^j_k}\right)\right)^l i_{\omega^\vee} \\
& \left(\left(\sum_{j=0}^{d(|\Gamma_0|-1)} \sum_{\substack{0\leqslant k\leqslant d'(|\Gamma_0|-1),\\ 0\leqslant(d+d')|\Gamma_1|-j-k\leqslant|\Gamma_1|}} P_{j,k}(\overline{w},\vec{x},d\overline{w},d\vec{x})\cdot\frac{\widetilde{\widetilde{P}}_{j,k}(d(\widetilde{\xi}))}{\widetilde{P}_{j,k}(\widetilde{\xi})}\right)\wedge\Phi\right) \\
=\quad & \Big|_{w^i=0,q^i=0,i\neq|\Gamma_0|}\,,
\end{aligned}
$$

where $i_{\omega^\vee}$ is the inner product of a differential form with a nonzero constant top polyvector field $\omega^\vee$ on $(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|-1}$, i.e. it maps a volume form to its coefficient.

We notice that for the top form part of above formula, the lowest power of $\widetilde{\rho}$ is

$$\begin{aligned}
& (2d+d')(|\Gamma_0|-|\Gamma_1|) - m(2d+d') + 2d|\Gamma_1| - 2j + d'|\Gamma_1| - k \\
= & (2d+d')|\Gamma_0| - m(2d+d') - 2j - k \\
= & (2d+d')|\Gamma_0| - m(2d+d') - j + |\Gamma_1| - 1 - (d+d')|\Gamma_1| \\
\geqslant & (2d+d')|\Gamma_0| - m(2d+d') - d(|\Gamma_0|-1) + |\Gamma_1| - 1 - (d+d')|\Gamma_1| \\
= & (d+d')|\Gamma_0| - (d+d'-1)|\Gamma_1| - m(d+d') - 1 - d(m-1),
\end{aligned}$$

so if Γ is not a (connected) Laman graph,

$$\int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),\Phi) = 0.$$

When Γ is indeed a Laman graph, we have

$$\int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),\Phi) = \int_{(\mathbf{R}^{d'}\times\mathbf{C}^d)_{\left(w^{|\Gamma_0|},q^{|\Gamma_0|}\right)}} (D\Phi)\Big|_{w^i=0,i\neq|\Gamma_0|},$$

where

$$\begin{aligned}
D = & (2\pi)^{\left(d+\frac{1}{2}d'\right)(|\Gamma_0|-m)} \int_{\partial C_\Gamma} \left(\frac{1}{\det\Big(M_\Gamma(\widetilde{\xi}^2)\Big)^{d+\frac{1}{2}d'} l_0!} \right. \\
& \left(\sum_{i=1}^{|\Gamma_0|-1}\sum_{j=1}^{|\Gamma_0|-1} \left(\sum_{k=1}^{d} \frac{1}{2} M_\Gamma(\widetilde{\xi}^2)^{-1}_{ij} \partial_{w^i_k}\partial_{\widetilde{w}^j_k} + \sum_{k=1}^{d'} M_\Gamma(\widetilde{\xi}^2)^{-1}_{ij} \partial_{q^i_k}\partial_{q^j_k} \right) \right)^{l_0} i_{\omega^\vee} \\
& \left. \left(\left(P_{d(|\Gamma_0|-1),d'(|\Gamma_0|-1)}(\overline{w},x,d\overline{w},dx) \cdot \frac{\widetilde{\widetilde{P}}_{\mathrm{jd}(|\Gamma_0|-1),d'(|\Gamma_0|-1)}(d(\widetilde{\xi}))}{\widetilde{P}_{d(|\Gamma_0|-1),d'(|\Gamma_0|-1)}(\widetilde{\xi})} \right) \wedge \Phi \right) \right|_{w^i=0,q^i=0,i\neq|\Gamma_0|},
\end{aligned}$$

where

$$l_0 = \left(\sum_{e\in\Gamma_1, 1\leqslant i\leqslant d} n_{i,e} \right) + \left(d + \frac{1}{2}d' \right)(|\Gamma_1| - |\Gamma_0| + 1).$$

In above formula of D, we can argue any terms that contain $\partial_{q^i_k}\Phi$ is zero: the order of

$$\left(\sum_{i=1}^{|\Gamma_0|-1}\sum_{j=1}^{|\Gamma_0|-1} \left(\sum_{k=1}^{d} \frac{1}{2} M_\Gamma(\widetilde{\xi}^2)^{-1}_{ij} \partial_{w^i_k}\partial_{\widetilde{w}^j_k} + \sum_{k=1}^{d'} M_\Gamma(\widetilde{\xi}^2)^{-1}_{ij} \partial_{q^i_k}\partial_{q^j_k} \right) \right)^{l_0}$$

is $2l_0$, to kill the variables $\{\overline{w}_i\}_{i\in\Gamma_0}$ in $P_{j,k}$, the order of each term with respect to $\{w_i, \overline{w}_i\}_{i\in\Gamma_0}$ is at least

$$2\left(\sum_{e\in\Gamma_1, 1\leqslant i\leqslant d} n_{i,e} + d(|\Gamma_1| - |\Gamma_0| + 1)\right).$$

So the order of each term with respect to $\{\vec{q}_i\}_{i\in\Gamma_0}$ is at most

$$d'(|\Gamma_1| - |\Gamma_0| + 1).$$

Note the degree of $P_{j,k}$ with respect to variables $\{\vec{q}_i\}_{i\in\Gamma_0}$ is also

$$d'(|\Gamma_1| - |\Gamma_0| + 1),$$

there's no additional orders left to act on Φ. $\square$

Now, we consider the general case. Given a subgraph $\Gamma' \subseteq \Gamma$, we use $\Gamma\backslash\Gamma'$ to denote the subgraph with the vertices Γ_0, and the edges $\Gamma_1\backslash\Gamma'_1$.[5] We have the following result:

Proposition 3.22. *Given a connected decorated graph (Γ, n) without self-loops, $\Phi \in \Omega^\bullet_c((\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|})$, and a subgraph Γ', we have*

(1) The integral

$$\begin{aligned}&\int_{\widetilde{[0,+\infty]}^{|\Gamma'_1|}/\mathbf{R}^+\times\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1\backslash\Gamma'_1|}} t^*_{\text{square}} \int_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma, n), \Phi)\\ =& \sum_{n'\in\mathrm{dec}(\Gamma/\Gamma')} C_{n'} W_0^L((\Gamma/\Gamma', n'), D_{n'}\Phi),\end{aligned}$$

where $\mathrm{dec}(\Gamma/\Gamma')$ is the set of all decorations of Γ/Γ', and $C_{n'} = 0$ for all but finite many $n' \in \mathrm{dec}(\Gamma/\Gamma')$, $D_{n'}$ are differential operators with constant coefficients, which only involve holomorphic derivatives.

(2) If $O_{(\Gamma', n|_{\Gamma'})}(\widetilde{\Phi}) = 0$ for any $\widetilde{\Phi} \in \Omega^\bullet_c((\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|})$,

$$\int_{\widetilde{[0,+\infty]}^{|\Gamma'_1|}/\mathbf{R}^+\times\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1\backslash\Gamma'_1|}} t^*_{\text{square}} \int_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma, n), \Phi) = 0.$$

(3) When $L \to +\infty$,

$$\lim_{L\to+\infty} \int_{\widetilde{[0,+\infty]}^{|\Gamma'_1|}/\mathbf{R}^+\times\widetilde{\left[0,\sqrt{L}\right]}^{|\Gamma_1\backslash\Gamma'_1|}} t^*_{\text{square}} \int_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|}} \widetilde{W}((\Gamma, n), -)$$

exists as a distribution.

[5] Note $\Gamma\backslash\Gamma'$ is different from Γ/Γ'.

Proof. From Proposition 3.21, we have

$$
\begin{aligned}
& \int\limits_{\widetilde{[0,+\infty]}^{|\Gamma_1'|}/\mathbf{R}^+ \times \widetilde{[0,\sqrt{L}]}^{|\Gamma_1\backslash\Gamma_1'|}} t^*_{\text{square}} \int\limits_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),\Phi) \\
= & \pm \int\limits_{\widetilde{[0,+\infty]}^{|\Gamma_1'|}/\mathbf{R}^+ \times \widetilde{[0,\sqrt{L}]}^{|\Gamma_1\backslash\Gamma_1'|}} t^*_{\text{square}} \int\limits_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \\
& \widetilde{W}((\Gamma',n|_{\Gamma'}),\widetilde{W}((\Gamma\backslash\Gamma',n|_{\Gamma\backslash\Gamma'}),\Phi)) \\
= & \pm \int\limits_{\widetilde{[0,\sqrt{L}]}^{|\Gamma_1\backslash\Gamma_1'|}} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|(\Gamma/\Gamma')_0|-1}} \int\limits_{\widetilde{[0,+\infty]}^{|\Gamma_1'|}/\mathbf{R}^+} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0'|}} t^*_{\text{square}} \\
& \widetilde{W}((\Gamma',n|_{\Gamma'}),\widetilde{W}((\Gamma\backslash\Gamma',n|_{\Gamma\backslash\Gamma'}),\Phi)) \\
= & \pm \int\limits_{\widetilde{[0,\sqrt{L}]}^{|\Gamma_1\backslash\Gamma_1'|}} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|(\Gamma/\Gamma')_0|-1}} \\
& O_{(\Gamma',n|_{\Gamma'})}(\widetilde{W}((\Gamma\backslash\Gamma',n|_{\Gamma\backslash\Gamma'}),\Phi)) \\
= & \pm \int\limits_{\widetilde{[0,\sqrt{L}]}^{|\Gamma_1\backslash\Gamma_1'|}} \int\limits_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|(\Gamma/\Gamma')_0|-1}} t^*_{\text{square}} \\
& (D\widetilde{W}((\Gamma\backslash\Gamma',n|_{\Gamma\backslash\Gamma'}),\Phi))\Big|_{w^i=0,q^i=0,i\neq|\Gamma_0|},
\end{aligned}
$$

where D is a differential operator with constant coefficients, which only involves holomorphic derivatives. Therefore, by Leibniz' rule, it is easy to see that

$$
\begin{aligned}
& \int\limits_{\widetilde{[0,+\infty]}^{|\Gamma_1'|}/\mathbf{R}^+ \times \widetilde{[0,\sqrt{L}]}^{|\Gamma_1\backslash\Gamma_1'|}} t^*_{\text{square}} \int\limits_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),\Phi) \\
= & \sum_{n'\in\operatorname{dec}(\Gamma/\Gamma')} C_{n'} W_0^L((\Gamma/\Gamma',n'),D_{n'}\Phi),
\end{aligned}
$$

where $C_{n'}=0$ for all but finitely many $n'\in\operatorname{dec}(\Gamma/\Gamma')$, $D_{n'}$ are differential operators with constant coefficients, which only involve holomorphic derivatives.

Other statements follow easily. □

3.3.3. *Anomaly vanishing results.* From last subsection, we know the anomalies are from local holomorphic functionals $O_{(\Gamma,n)}$. In this section, we prove two vanishing results for $O_{(\Gamma,n)}$.

Proposition 3.23. *Given a connected decorated Laman graph (Γ, n) without self-loops, and $\Phi \in \Omega_c^\bullet((\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|})$. If $d' \geqslant 1$ and the 1st Betti number $h_1(\Gamma) = \dim(H_1(\Gamma))$ is odd, we have*

$$O_{(\Gamma,n)}(\Phi) = 0.$$

Proof. Let $H \in \mathbf{R}^{d'}$ be a hyperplane passing through the origin, we choose a unit normal vector of H and denote it by $\vec{n}$. Then we define a map $r : (\mathbf{R}^{d'})^{|\Gamma_0|} \to (\mathbf{R}^{d'})^{|\Gamma_0|}$ by the following formula:

$$\begin{aligned} & r\left(q^1, q^2, \ldots, q^{|\Gamma_0|-1}, q^{|\Gamma_0|}\right) \\ = \quad & \left(q^1 - 2(q^1 \cdot \vec{n})\vec{n}, q^2 - 2(q^2 \cdot \vec{n})\vec{n}, \ldots, q^{|\Gamma_0|-1} - 2\left(q^{|\Gamma_0|-1} \cdot \vec{n}\right)\vec{n}, q^{|\Gamma_0|}\right), \end{aligned}$$

i.e., r is the reflection with respect to a hyperplane through $x^{|\Gamma_0|}$. From (43) we can get

$$r^* \widetilde{W}((\Gamma, n), \Phi) = (-1)^{|\Gamma_1|} \widetilde{W}((\Gamma, n), r^*\Phi).$$

Note r will change the orientation of $(\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|}$ by a $(-1)^{1-|\Gamma_0|}$ factor, we have

$$\begin{aligned} O_{(\Gamma,n)}(\Phi) &= \int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|}} \widetilde{W}((\Gamma, n), \Phi) \\ &= (-1)^{|\Gamma_1| - |\Gamma_0| + 1} \int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{C}^d \times \mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma, n), r^*\Phi) \\ &= (-1)^{h_1(\Gamma)} O_{(\Gamma,n)}(r^*\Phi) \\ &= (-1)^{h_1(\Gamma)} O_{(\Gamma,n)}(\Phi) \end{aligned}$$

In the last step, we used Proposition 3.21: since $O_{(\Gamma,n)}$ is a local holomorphic functional, it only depends on the values of Φ and holomorphic derivatives on the diagonal. However, Φ and $r^*\Phi$ have the same values and holomorphic derivatives on the diagonal.

Now, it is obvious that $O_{(\Gamma,n)}(\Phi) = 0$ if $h_1(\Gamma)$ is odd. □

Proposition 3.24. *Given a connected decorated Laman graph (Γ, n) without self-loops, and $\Phi \in \Omega_c^\bullet((\mathbf{R}^{d'} \times \mathbf{C}^d)^{|\Gamma_0|})$. If $d' \geqslant 2$ and $|\Gamma_0| \geqslant 3$, we have*

$$O_{(\Gamma,n)}(\Phi) = 0.$$

Proof. Let's first consider the case when $d + d' \geqslant 3$.

We prove any vertice of Laman graph is at least bivalent. If $v \in \Gamma_0$ is univalent and $e \in \Gamma_1$ conneccts v, we consider the subgraph Γ' given by

$$\Gamma'_0 = \Gamma_0 \backslash \{v\}, \Gamma'_1 = \Gamma_1 \backslash \{e\}.$$

Since Γ is Laman and $|\Gamma'_0| = |\Gamma_0| - 1 \geqslant 2$, we have

$$\begin{aligned} & (d+d')|\Gamma'_0| \\ = \ & (d+d')|\Gamma_0| - d - d' \\ = \ & (d+d'-1)|\Gamma_1| + 1 \\ = \ & (d+d'-1)|\Gamma'_1| + d + d' \\ \leqslant \ & (d+d'-1)|\Gamma'_1| + d + d' + 1. \end{aligned}$$

This contradicts (71).

Then we prove the existence of a bivalent vertex in Γ_0. If there's no bivalent vertices, each vertex must be at least trivalent, so

$$\frac{3}{2}|\Gamma_0| \leqslant |\Gamma_1| = \frac{(d+d')|\Gamma_0| - d - d' - 1}{d+d'-1},$$

therefore

$$0 \leqslant \frac{d+d'-3}{2}|\Gamma_0| \leqslant -d - d' - 1 < 0,$$

we get a contradiction. So, there exists a bivalent vertex v. We use v_1 and v_2 to denote the two vertices adjacent to v. Without loss of generality, we assume $v \neq |\Gamma_0|$. By Proposition 3.21, we have

$$O_{(\Gamma,n)}(\Phi) = O_{(\Gamma,n)}\left(\Phi\Big|_{q^i=0, i\neq|\Gamma_0|}\right),$$

we only need to prove

$$O_{(\Gamma,n)}\left(\Phi\Big|_{q^i=0, i\neq|\Gamma_0|}\right) = 0.$$

For fixed $\left(q^1, q^2, \ldots, q^{v-1}, \widehat{q^v}, q^{v+1}, \ldots, q^{|\Gamma_0|}\right) \in (\mathbf{R}^{d'})^{|\Gamma_0|-1}$, we choose a hyperplane H that pass through v_1 and v_2(this is possible when $d' \geqslant 2$), We denote the reflection map of vertex v by r_v. Then we have

$$
\begin{aligned}
& \int_{\mathbf{R}^{d'}_{q^v}} \widetilde{W}\left((\Gamma,n),\Phi\Big|_{q^i=0,i\neq|\Gamma_0|}\right) \\
= & \int_{\mathbf{R}^{d'}_{q^v}} r_v^*\widetilde{W}\left((\Gamma,n),\Phi\Big|_{q^i=0,i\neq|\Gamma_0|}\right) \\
= & -\int_{\mathbf{R}^{d'}_{q^v}} \widetilde{W}\left((\Gamma,n),\Phi\Big|_{q^i=0,i\neq|\Gamma_0|}\right).
\end{aligned}
$$

So

$$
\begin{aligned}
& O_{(\Gamma,n)}(\Phi) \\
= & O_{(\Gamma,n)}\left(\Phi\Big|_{q^i=0,i\neq|\Gamma_0|}\right) \\
= & \pm\int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{R}^{d'}\times\mathbf{C}^d)^{|\Gamma_0|-1}} \int_{\mathbf{C}^d_{w^v}} \int_{\mathbf{R}^{d'}_{q^v}} \widetilde{W}\left((\Gamma,n),\Phi\Big|_{q^i=0,i\neq|\Gamma_0|}\right) \\
= & 0.
\end{aligned}
$$

Finally, we deal with the case when $d=0$ and $d'=2$. In this case, by Proposition 3.21, we have

$$
\begin{aligned}
& O_{(\Gamma,n)}(\Phi) \\
= & \pm\int_{\mathbf{R}^2_{q^{|\Gamma_0|}}} \Phi\Big|_{q^i=0,i\neq|\Gamma_0|} \int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{R}^2)^{|\Gamma_0|-1}} \widetilde{W}((\Gamma,n),1).
\end{aligned}
$$

Therefore, we only need to prove

$$
\int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{R}^2)^{|\Gamma_0|-1}} \widetilde{W}((\Gamma,n),1) = 0.
$$

By Lemma 3.12,

$$
t^*_{\text{square}} \int_{(\mathbf{R}^2)^{|\Gamma_0|-1}} \widetilde{W}((\Gamma,n),1)
$$

is invariant unnder the scaling transformation, so it is not hard to prove

$$
\int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{R}^2)^{|\Gamma_0|-1}} \widetilde{W}((\Gamma,n),1) = \int_{S^{|\Gamma_1|}} t^*_{\text{square}} \int_{(\mathbf{R}^2)^{|\Gamma_0|-1}} \widetilde{W}((\Gamma,n),1),
$$

where

$$S^{|\Gamma_1|} = \left\{ (\widetilde{t}_e)_e \in [0,+\infty)^{|\Gamma_1|} \middle| \sum_{e \in \Gamma_1} \widetilde{t}^2 = 1 \right\}.$$

Let

$$S^{2|\Gamma_0|-3} = \left\{ (q^i)_{i \neq |\Gamma_0|} \in (\mathbf{R}^2)^{|\Gamma_0|-1} | \sum_{i \neq |\Gamma_0|,} |q^i|^2 = 1 \right\}.$$

Note

$$S^{|\Gamma_1|} \times (\mathbf{R}^2)^{|\Gamma_0|-1} \to ([0,+\infty)^{|\Gamma_1|} \times (\mathbf{R}^2)^{|\Gamma_0|-1})/\mathbf{R}^+$$

and

$$[0,+\infty)^{|\Gamma_1|} \times S^{2|\Gamma_0|-3} \to ([0,+\infty)^{|\Gamma_1|} \times (\mathbf{R}^2)^{|\Gamma_0|-1})/\mathbf{R}^+$$

are open dense submanifolds of $([0,+\infty)^{|\Gamma_1|} \times (\mathbf{R}^2)^{|\Gamma_0|-1})/\mathbf{R}^+$, we have

$$\begin{aligned}
& \int_{\partial C_\Gamma} t^*_{\text{square}} \int_{(\mathbf{R}^2)^{|\Gamma_0|-1}} \widetilde{W}((\Gamma,n),1) \\
= & \int_{S^{|\Gamma_1|}} t^*_{\text{square}} \int_{(\mathbf{R}^2)^{|\Gamma_0|-1}} \widetilde{W}((\Gamma,n),1) \\
= & \int_{([0,+\infty)^{|\Gamma_1|} \times (\mathbf{R}^2)^{|\Gamma_0|-1})/\mathbf{R}^+} t^*_{\text{square}} \widetilde{W}((\Gamma,n),1) \\
= & \int_{[0,+\infty)^{|\Gamma_1|} \times S^{2|\Gamma_0|-3}} t^*_{\text{square}} \widetilde{W}((\Gamma,n),1) \\
= & \int_{\mathrm{Conf}(\Gamma)/(\mathbf{R}^+ \ltimes \mathbf{R}^2)} \prod_{e \in \Gamma_1} \widetilde{P}_{0,+\infty}(x^{h(e)} - x^{t(e)}) \\
= & \; 0,
\end{aligned}$$

where $\mathbf{R}^+ \ltimes \mathbf{R}^2$ is the affine transformation on $\mathrm{Conf}(\Gamma)$. In the last step, we used Kontsevich's lemma in [Kon03, Lemma 6.4]. This finishes our proof. □

Remark 3.25. The above lemma is a direct generalization of some results of Kontsevich in his work on the renormalization of perturbative topological field theories, see [Kon03; Kon94].

Finally, we have arrived our main theorem on the existence of quantization for topological-holomorphic theories:

Theorem 3.26. *If $d' > 1$, and the classical master equation of topological-holomorphic theories on $\mathbf{R}^{d'} \times \mathbf{C}^d$ holds, then the quantum master equation is satisfied. In particular, the observables of the resulting quantum topological-holomorphic theory defines a factorization algebra on $\mathbf{R}^{d'} \times \mathbf{C}^d$.*

Proof. By Corollary 2.7, we only need to show the following equality for stable graph γ.

$$\begin{aligned}(69)\qquad & \sum_{e\in E(\gamma)} w_{\gamma,e}(P_{0<L}, Q(P_{0<L}), I) + \sum_{e\in E(\gamma)} \lim_{t\to 0} \sum_{e\in E(\gamma)} w_{\gamma,e}(P_{0<L}, K_t, I) \\ + \;& w_{\gamma,e}(P_{0<L}, K_L, I) \\ = \;& 0.\end{aligned}$$

On the other hand, by equation (67), we have

$$\begin{aligned}(70)\qquad & ((\bar\partial + d)W_0^L)((\Gamma,n),-) \pm t^*_{\text{square}} \int_{\partial_L \widetilde{[0,\sqrt{L}]}^{|\Gamma_1|}} \int_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),-) \\ \pm \;& t^*_{\text{square}} \int_{\partial_0 \widetilde{[0,\sqrt{L}]}^{|\Gamma_1|}} \int_{(\mathbf{C}^d\times\mathbf{R}^{d'})^{|\Gamma_0|}} \widetilde{W}((\Gamma,n),-) \\ = \;& 0,\end{aligned}$$

If we expand the expression of the first term in (69), we will find out it is corresponds to the first term in (70). Likewise, the second term in (69) corresponds to the second term in (70).

Finally, by Proposition 3.24, the third term in (70) only have contributions from contracting edges, so it corresponds to the third term in (69). This completes the proof of quantum master equation. The existence of factorization algebra follows from [CG21]. □

As a corollary we obtain the following result about well-studied topological-holomorphic gauge theories.

Corollary 3.27. *Let $\mathfrak{g}$ be a Lie algebra which is equipped with an invariant, non-degenerate, symmetric pairing. Then, for $d' > 1$, hybrid Chern–Simons theory on $\mathbf{R}^{d'} \times \mathbf{C}^d$ with gauge Lie algebra $\mathfrak{g}$ admits a quantization, hence factorization algebra of observables, to all orders in perturbation theory.*

In particular, consider the list of holomorphic topological theories given at the end of section 2.2. The theories (1),(2),(6),(7),(8),(9) admit quantizations to all orders in perturbation theory for any gauge Lie algebra

Remark 3.28. The existence of the quantization of ordinary Chern–Simons theory is well-known. The existence of the quantization of (2) has already been proven in the work that the theory was introduced [Cos13b].

APPENDIX A. GRAPH THEORY BACKGROUND

In this appendix, we introduce some concepts and facts from graph theory. These facts will be used in the study of Feynman graph integrals.

Definition A.1. A **directed graph** Γ consists of the following data:

1. A set of vertices Γ_0 and a set of edges Γ_1.
2. An ordering of all the vertices Γ_0 and an ordering of all the edges Γ_1.
3. Two maps
$$t, h : \Gamma_1 \to \Gamma_0$$
which are the assignments of tail and head to each directed edge.
Furthermore, we say Γ is **decorated** in $\mathbf{C}^d \times \mathbf{R}^{d'}$ if we have a map
$$n : e \in \Gamma_1 \to (n_{1,e}, n_{2,e}, \dots, n_{d,e}) \in (\mathbb{Z}^{\geqslant 0})^d,$$
which associates each edge d non-negative integer. We use (Γ, n) to denote a decorated graph. If n is the zero map, we will simply write Γ for (Γ, n). We say Γ is a **graph without self-loops**, if $t(e) \neq h(e)$ for any $e \in \Gamma_1$.

We will use $|\Gamma_0|$ and $|\Gamma_1|$ to denote the number of vertices and edges respectively.

Definition A.2. Given a connected directed graph Γ without self-loops, we have the following **incidence matrix**:
$$\rho_i^e = \begin{cases} 1 & \text{if } h(e) = i \\ -1 & \text{if } t(e) = i \\ 0 & \text{otherwise} \end{cases}$$
where $i \in \Gamma_0$, $e \in \Gamma_1$.

Definition A.3. Given a connected directed graph Γ, a tree $T \subseteq \Gamma$ is said to be a spanning tree if every vertex of Γ lies in T. We denote the set of all spanning tree by $\mathrm{Tree}(\Gamma)$.

Definition A.4. Given a connected directed graph Γ without self-loops and two disjoint subsets of vertices $V_1,\ V_2 \subseteq \Gamma_0$, we define $\mathrm{Cut}(\Gamma; V_1, V_2)$ to be the set of subsets $C \subseteq \Gamma_1$ satisfying the following properties:

1. The removing of edges in C from Γ divides Γ into exactly two connected trees, which we denoted by $\Gamma'(C),\ \Gamma''(C)$, such that $V_1 \subseteq \Gamma_0'(C),\ V_2 \subseteq \Gamma_0''(C)$.
2. C doesn't contain any proper subset satisfying property 1.

Definition A.5. Given a connected directed graph Γ without self-loops, and a function maps each $e \in \Gamma_1$ to $t_e \in (0, +\infty)$, we define the **weighted laplacian** of Γ by the following formula:

$$M_\Gamma(t)_{ij} = \sum_{e\in\Gamma_1} \rho_i^e \frac{1}{t_e} \rho_j^e, \quad 1 \leqslant i, j \leqslant |\Gamma_0| - 1.$$

The following facts will be used to show the finitenes of Feynman graph integrals.

Theorem A.6 (Kirchhoff)**.** *Given a connected graph Γ without self-loops, the determinant of weighted laplacian is given by the following formula:*

$$\det M_\Gamma(t) = \frac{\sum\limits_{T\in\mathrm{Tree}(\Gamma)} \prod\limits_{e\notin T} t_e}{\prod\limits_{e\in\Gamma_1} t_e}$$

Corollary A.7. *The inverse of $M_\Gamma(t)$ is given by the following formula:*

$$(M_\Gamma(t)^{-1})^{ij} = \frac{1}{\sum\limits_{T\in\mathrm{Tree}(\Gamma)} \prod\limits_{e\notin T} t_e} \cdot \left(\sum_{C\in\mathrm{Cut}(\Gamma;\{i,j\},\{|\Gamma_0|\})} \prod_{e\in C} t_e \right)$$

Corollary A.8. *We have the following equality:*

$$\begin{aligned} &\frac{1}{t_e} \sum_{i=1}^{|\Gamma_0|-1} \rho_i^e (M_\Gamma(t)^{-1})^{ij} \\ = \quad & \frac{1}{\sum\limits_{T\in\mathrm{Tree}(\Gamma)} \prod\limits_{e\notin T} t_e} \left(\sum_{C\in\mathrm{Cut}(\Gamma;\{j,h(e)\},\{|\Gamma_0|,t(e)\})} \frac{\prod_{e'\in C} t_{e'}}{t_e} \right. \\ - \quad & \left. \sum_{C\in\mathrm{Cut}(\Gamma;\{j,t(e)\},\{|\Gamma_0|,h(e)\})} \frac{\prod_{e'\in C} t_{e'}}{t_e} \right) \end{aligned}$$

In particular, every term of the numerator also appears in the denominator, so we have

$$\left| \frac{1}{t_e} \sum_{i=1}^{|\Gamma_0|-1} \rho_i^e (M_\Gamma(t)^{-1})^{ij} \right| \leqslant 2.$$

We use $(d_\Gamma(t)^{-1})^{ej}$ to denote $\frac{1}{t_e}\sum_{i=1}^{|\Gamma_0|-1}\rho_i^e(M_\Gamma(t)^{-1})^{ij}$.

Proof. See [Li12, Appendix B]. □

Remark A.9. If we view graphs as discrete spaces, the incidence matrix can be viewed as de Rham differential. Then $(d_\Gamma(t)^{-1})^{ej}$ can be viewed as the Green's function of de Rham differential on a graph. This might explains the importance of $(d_\Gamma(t)^{-1})^{ej}$.

Finally, we introduce the concept of Laman graphs, which will be useful to describe anomalies of Feynman graph integrals.

Definition A.10. Given a connected directed graph Γ without self-loops, we call Γ a Laman graph for $\mathbf{C}^d\times\mathbf{R}^{d'}$, if the following conditions hold:

1. For any subgraph Γ' such that $|\Gamma_0'|\geqslant 2$, we have the following inequality:

$$(d+d')|\Gamma_0'|\geqslant(d+d'-1)|\Gamma_1'|+d+d'+1. \tag{71}$$

2. The following equality holds:

$$(d+d')|\Gamma_0|=(d+d'-1)|\Gamma_1|+d+d'+1. \tag{72}$$

We use the notation $\mathrm{Laman}(\Gamma)$ to denote all the Laman subgraphs of Γ.

Remark A.11. When $d+d'=2$, the concept of Laman graph originate from Laman's characterization of generic rigidity of graphs embeded in two dimensional real linear space(see [Lam70]). Our definition appears in [Bud+23].

Appendix B. Compactification of Schwinger spaces

In this appendix, we review the construction of a compactification of Schwinger space in [Wan24].

Definition B.1. Given a directed graph Γ, and $L>0$, the **Schwinger space** is defined by $(0,L]^{|\Gamma_1|}$. The orientation is given by the following formula:

$$\int_{(0,L]^{|\Gamma_1|}}\prod_{e\in\Gamma_1}dt_e=L^{|\Gamma_1|}.$$

Assume Γ is a connected directed graph, $L>0$ is a positive real number. Let $S\subseteq\Gamma_1$ be a subset of Γ_1, we define the following submanifold with corners of Schwinger space:

$$\Delta_S=\left\{(t_1,t_2,\dots,t_{|\Gamma_1|})\in[0,L]^{|\Gamma_1|}|t_e=0\quad\text{if } e\in S\right\}.$$

The compactification of Schwinger space is obtained by iterated real blow ups of $[0, L]^{|\Gamma_1|}$ along Δ_S for all $S \subseteq \Gamma_1$ in a certain order (see [Kön21; AMN22]). To avoid getting into technical details of real blow ups of manifolds with corners, we will use another simpler definition. Instead, we present a typical example of real blow up, which will be helpful to understand our construction:

Example B.2. Let $S = \{1, 2, \ldots, k\} \subseteq \Gamma_1$, the real blow up of $[0, +\infty)^{|\Gamma_1|}$ along Δ_S is the following manifold(with corners):

$$[[0, +\infty)^{|\Gamma_1|} : \Delta_S] := \left\{ (\rho, \xi_1, \ldots, \xi_k, t_{k+1}, \ldots t_{|\Gamma_1|}) \in [0, +\infty)^{|\Gamma_1|+1} \middle| \sum_{i=1}^{k} \xi_i^2 = 1 \right\}.$$

We have a natural map from $[[0, +\infty)^{|\Gamma_1|} : \Delta_S]$ to $[0, +\infty)^{|\Gamma_1|}$:

$$(\rho, \xi_1, \ldots, \xi_k, t_{k+1}, \ldots t_{|\Gamma_1|}) \to (t_1 = \rho\xi_1, \ldots, t_k = \rho\xi_k, t_{k+1}, \ldots, t_{|\Gamma_1|}).$$

We also have a natural inclusion map from $(0, +\infty)^{|\Gamma_1|}$ to $[[0, +\infty)^{|\Gamma_1|} : \Delta_S]$:

$$\begin{aligned} &(t_1, \ldots, t_{|\Gamma_1|}) \\ \to \quad &\left(\rho = \sqrt{\sum_{i=1}^{k} t_i^2}, \xi_1 = \frac{t_1}{\sqrt{\sum_{i=1}^{k} t_i^2}}, \ldots, \xi_k = \frac{t_k}{\sqrt{\sum_{i=1}^{k} t_i^2}}, t_{k+1}, \ldots, t_{|\Gamma_1|} \right). \end{aligned}$$

For us, the most important property is that $\frac{t_i}{\sqrt{\sum_{i=1}^{k} t_i^2}}$ can be extended to a smooth function ξ_i on $[[0, +\infty)^{|\Gamma_1|} : \Delta_S]$.

Let's consider the following natural inclusion map:

$$i : (0, +\infty)^{|\Gamma_1|} \to \prod_{S \subseteq \Gamma_1} [[0, +\infty)^{|\Gamma_1|} : \Delta_S]$$

Definition B.3. We call the closure of the image of $(0, L]^{|\Gamma_1|}$ under i the **compactified Schwinger space** of Γ. We denote it by $\widetilde{[0, L]}^{|\Gamma_1|}$.

Remark B.4. Sometimes, by abusing of concept, we will also call the closure of the image of $(0, +\infty)^{|\Gamma_1|}$ under i the compactified Schwinger space, although it is not compact. We will use $\widetilde{[0, +\infty)}^{|\Gamma_1|}$ to denote it.

Proposition B.5. $\widetilde{[0, L]}^{|\Gamma_1|}$ *is a compact manifold with corners.*

Proof. See [AMN22]. □

To obtain a more concrete description of $\widetilde{[0,L]^{|\Gamma_1|}}$, we introduce a useful concept called corners of compactified Schwinger spaces.

For $\Gamma_1 = S_0 \supseteq S_1 \supsetneq S_2 \supsetneq \cdots \supsetneq S_m \supsetneq S_{m+1} = \varnothing$, we define a submanifold

$$C_{S_1,S_2,\dots,S_m} \subseteq [0,+\infty)^m \times (0,+\infty)^{|S_0|-|S_1|} \times \prod_{i=1}^{m} (0,+\infty)^{|S_i|-|S_{i+1}|},$$

which is given by a set of equations:

if we use $\{\rho_i\}(1 \leqslant i \leqslant m)$ for the coordinates of the first product component $[0,+\infty)^m$, use $\{t_e\}(e \in S_0 \backslash S_1)$ for the coordinates of the product component $(0,+\infty)^{|S_0|-|S_1|}$, use $\{\xi_e^i\}$ $(1 \leqslant i \leqslant m, e \in S_i \backslash S_{i+1})$ for the coordinates of the product component $(0,+\infty)^{|S_{i-1}|-|S_i|}$, then:

$$\begin{cases} \sum\limits_{e\in S_m} (\xi_e^m)^2 = 1 \\ \sum\limits_{e\in S_{m-1}\backslash S_m} (\xi_e^{m-1})^2 + \sum\limits_{e\in S_m} (\rho_m \xi_e^m)^2 = 1 \\ \sum\limits_{e\in S_{m-2}\backslash S_{m-1}} (\xi_e^{m-2})^2 + \sum\limits_{e\in S_{m-1}\backslash S_m} (\rho_{m-1}\xi_e^{m-1})^2 + \sum\limits_{e\in S_m} (\rho_{m-1}\rho_m \xi_e^m)^2 = 1 \\ \cdots \\ \sum\limits_{i=1}^{k} \left(\sum\limits_{e\in S_{m-k+i}\backslash S_{m-k+i+1}} \left(\left(\prod\limits_{j=1}^{i-1} \rho_{m-k+j} \right) \xi_e^{m-k+i} \right)^2 \right) = 1 \\ \cdots \\ \sum\limits_{i=1}^{m} \left(\sum\limits_{e\in S_{m-k+i}\backslash S_{m-k+i+1}} \left(\left(\prod\limits_{j=1}^{i-1} \rho_{m-k+j} \right) \xi_e^{m-k+i} \right)^2 \right) = 1 \end{cases}$$

We have a natural map from $C_{S_1,S_2,\dots,S_m}$ to $[0,+\infty)^{|\Gamma_1|}$: Let $\{\widetilde{t_e}\}_{e\in\Gamma_1}$ be the coordinates of $[0,+\infty)^{|\Gamma_1|}$, then under the natural map, we have:

$$\begin{cases} \widetilde{t_e} = t_e \quad \text{for } e \in S_0 \backslash S_1, \\ \widetilde{t_e} = \rho_1 \xi_e^1 \quad \text{for } e \in S_1 \backslash S_2, \\ \widetilde{t_e} = \rho_1\rho_2 \xi_e^2 \quad \text{for } e \in S_2 \backslash S_3, \\ \cdots \\ \widetilde{t_e} = \prod\limits_{k=1}^{m} \rho_k \xi_e^m \quad \text{for } e \in S_m \backslash S_{m+1}. \end{cases}$$

Proposition B.6. *The following statements are true:*

1. *There is a natural inclusion map from* $C_{S_1,S_2,\dots,S_m}$ *to* $\widetilde{[0,+\infty)^{|\Gamma_1|}}$.

2. $\widetilde{[0,+\infty)}^{|\Gamma_1|}$ *is covered by the union of all* $C_{S_1,S_2,\dots,S_m}$*:*

$$\widetilde{[0,+\infty)}^{|\Gamma_1|} = \bigcup_{\Gamma_1 = S_0 \supseteq S_1 \supsetneq \cdots \supsetneq S_m \supsetneq S_{m+1} = \varnothing} C_{S_1,S_2,\dots,S_m}.$$

3. *There is a natural action of* $\mathbf{R}_+ = (0,+\infty)$ *on* $\widetilde{[0,+\infty)}^{|\Gamma_1|}$*, which has the following form on* $(0,+\infty)^{|\Gamma_1|} \subseteq \widetilde{[0,+\infty)}^{|\Gamma_1|}$*:*

$$\lambda \cdot (t_e)_{e\in\Gamma_1} = (\lambda t_e)_{e\in\Gamma_1} \quad \lambda \in (0,+\infty).$$

Proof. Left as an excercise for the reader. □

Definition B.7. For $\Gamma_1 = S_0 \supseteq S_1 \supsetneq S_2 \supsetneq \cdots \supsetneq S_m \supsetneq S_{m+1} = \varnothing$, let Γ'^i be the subgraph generated by $S_i (1 \leqslant i \leqslant m)$. We call $C_{S_1,S_2,\dots,S_m} \subseteq \widetilde{[0,+\infty)}^{|\Gamma_1|}$ the **corner of compactified Schwinger space** corresponds to $\Gamma'^1, \Gamma'^2, \dots, \Gamma'^m \to 0$.

We will use $\partial C_{S_1,S_2,\dots,S_m}$ to denote the boundary of $C_{S_1,S_2,\dots,S_m}$. As a set, it is

$$\{p \in C_{S_1,S_2,\dots,S_m} | \rho_i(p) = 0 \text{ for } 1 \leqslant i \leqslant m\}.$$

Remark B.8. Note $\widetilde{[0,+\infty)}^{|\Gamma_1|}$ is a manifold with corners. In particular, it is a stratified space. Its codimension m strata is

$$\bigcup_{\Gamma_1 = S_0 \supseteq S_1 \supsetneq \cdots \supsetneq S_m \supsetneq S_{m+1} = \varnothing} \partial C_{S_1,S_2,\dots,S_m}.$$

The matrice $(M_\Gamma(t)^{-1})^{ij}$ and $(d_\Gamma(t)^{-1})^{ej}$ defined in Appendix A can be extended to smooth functions on $\widetilde{[0,+\infty)}^{|\Gamma_1|}$:

Lemma B.9. *Given a connected graph* Γ *without self-loops, The following functions can be extended to smooth functions on* $\widetilde{[0,+\infty)}^{|\Gamma_1|}$*:*

1. $(M_\Gamma(t)^{-1})^{ij}$ *for* $1 \leqslant i,j \leqslant |\Gamma_0| - 1$.
2. $(d_\Gamma(t)^{-1})^{ej}$ *for* $e \in \Gamma_1$, $1 \leqslant j \leqslant |\Gamma_0| - 1$.

Proof. See [Wan24]. □

The following lemma will be used in the prove of finiteness of Feynman graph integrals for $\mathbf{R}^{d'}$.

Lemma B.10. *The map*

$$(t_e)_{e\in\Gamma_1} \in (0,+\infty)^{|\Gamma_1|} \to (t_e^2)_{e\in\Gamma_1} \in (0,+\infty)^{|\Gamma_1|}$$

can be extended to a smooth map t_{square} *from* $\widetilde{[0,+\infty)}^{|\Gamma_1|}$ *to* $\widetilde{[0,+\infty)}^{|\Gamma_1|}$.

Proof. From Proposition B.6, we only need to extend the map to a map from $C_{S_1,S_2,\ldots,S_m}$ to $C_{S_1,S_2,\ldots,S_m}$. Let's prove that this map can be extended to $C_{\Gamma_1'}$, where Γ_1' is a subgraph of Γ. The general situation is left to the reader.

This can be shown by using the coordinates $\{\rho, t_e, \xi_{e'}\}_{e\in\Gamma_1\backslash\Gamma_1', e'\in\Gamma_1'}$ we have introduced. More precisely, $\{\rho, t_e, \xi_{e'}\}_{e\in\Gamma_1\backslash\Gamma_1', e'\in\Gamma_1'}$ will be mapped to $\{\tilde{\rho}, \widetilde{t_e}, \widetilde{\xi_{e'}}\}_{e\in\Gamma_1\backslash\Gamma_1', e'\in\Gamma_1'}$, such that

$$\begin{cases} \widetilde{t_e} = t_e^2 \quad e \in \Gamma_1\backslash\Gamma_1', \\ \widetilde{\xi_{e'}} = \frac{\xi_{e'}^2}{\sqrt{\sum\limits_{e'\in\Gamma_1'} \xi_{e'}^4}} \quad e \in \Gamma_1', \\ \tilde{\rho} = \rho^2 \sqrt{\sum\limits_{e'\in\Gamma_1'} \xi_{e'}^4}. \end{cases}$$

It is clear that this map is smooth on $C_{\Gamma_1'}$. □

Finally, we give a description of the boundary of compactified Schwinger space:

Proposition B.11. *Given a connected graph Γ without self-loops, the boundary $\partial\widetilde{[0,L]^{|\Gamma_1|}}$ of $\widetilde{[0,L]^{|\Gamma_1|}}$ has the following decomposition:*

$$\partial\widetilde{[0,L]^{|\Gamma_1|}} = \left(-\partial_0\widetilde{[0,L]^{|\Gamma_1|}}\right) \cup \partial_L\widetilde{[0,L]^{|\Gamma_1|}},$$

where $\partial_0\widetilde{[0,L]^{|\Gamma_1|}}$($\partial_L\widetilde{[0,L]^{|\Gamma_1|}}$) describe the boundary components near the origin(away from the origin). More precisely, we have

$$\begin{cases} \partial_0\widetilde{[0,L]^{|\Gamma_1|}} = \bigcup_{\Gamma'\subseteq\Gamma} (-1)^{\sigma(\Gamma',\Gamma/\Gamma')} \widetilde{[0,+\infty]^{|\Gamma_1'|}}/\mathbf{R}^+ \times \widetilde{[0,+L]^{|\Gamma_1\backslash\Gamma_1'|}} \\ \partial_L\widetilde{[0,L]^{|\Gamma_1|}} = \bigcup_{e\in\Gamma_1} (-1)^{|e|}\{L\} \times \widetilde{[0,L]^{|(\Gamma/e)_1|}} \end{cases}.$$

Proof. This follows from Proposition B.6 and the explicit description of $C_{S_1,S_2,\ldots,S_m}$. □

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