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The Virasoro vertex algebra and factorization algebras on Riemann surfaces

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Abstract This paper focuses on the connection of holomorphic two-dimensional factorization algebras and vertex algebras which has been made precise in the forthcoming book of Costello–Gwilliam. We provide a construction of the Virasoro vertex algebra starting from a local Lie algebra on the complex plane. Moreover, we discuss an extension of this factorization algebra to a factorization algebra on the category of Riemann surfaces. The factorization homology of this factorization algebra is computed as the correlation functions. We provide an example of how the Virasoro factorization algebra implements conformal symmetry of the beta–gamma system using the method of effective BV quantization.

Keywords Factorization algebras · Virasoro vertex algebra · BV quantization · Conformal field theory

Mathematics Subject Classification 81R10 · 17B65 · 18G55

1 Introduction

In this paper, we study the sheaf of holomorphic vector fields in one complex dimension and local extensions thereof. Using the formalism of factorization algebras developed in the book [9], we provide a construction of the *Virasoro factorization algebra* defined on any Riemann surface. Moreover, we compute and recognize the factorization homology of the two-dimensional factorization algebra as encoding the conformal blocks of the Virasoro vertex algebra.

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The Virasoro Lie algebra Vir arises as a central extension of the Lie algebra of vector fields on a circle $\mathrm{Vect}(S^1)$. In fact, it is the unique central extension as $H^2(\mathrm{Vect}(S^1))$ is one-dimensional with generator given by the Gelfand–Fuks cocycle [14] defined by

$$\omega_{\mathrm{GF}}(f(t)\partial_t, g(t)\partial_t) \mapsto \frac{1}{12}\int_{S^1} f'''(t)g(t)\mathrm{d}t.$$

The Virasoro Lie algebra, along with its related vertex algebra and category of representations, is interesting and natural in their own right from a mathematical point of view [11,14–16].

The compelling motivation for studying of the Virasoro algebra derived from understanding the symmetries of two-dimensional conformal field theories. Classically, conformal symmetry consists of two copies of the complexification of the Lie algebra of vector fields on the circle: a holomorphic and an anti-holomorphic version. We will choose to focus on holomorphic, or chiral, conformal field theories and hence only consider holomorphic vector fields on two-dimensional complex manifolds. The Weyl anomaly arises when one tries to quantize the symmetry of holomorphic vector fields on such a conformal field theory. It results in the one-dimensional central extension of holomorphic vector fields defined by the Gelfand–Fuks cocycle. Moreover, the anomaly is characterized by how the central parameter acts on the quantum theory; this is called the *central charge* of the theory.

We work with the Dolbeault resolution of holomorphic vector fields on $\mathbf{C}$, which we denote by $\mathcal{L}^{\mathbf{C}}$ throughout. The fact that we can restrict vector fields to open sets gives this the structure of a sheaf of Lie algebras. Moreover, it has the structure of a *local Lie algebra* on $\mathbf{C}$ which will be central in our construction. We define an explicit cocycle ω that defines a (-1)-shifted central extension of this local Lie algebra. There is a factorization algebra associated to this local Lie algebra, denoted $\mathcal{V}\mathrm{ir}$. We show that the factorization product encodes the product on the universal enveloping algebra associated to the ordinary Virasoro Lie algebra, $U(\mathrm{Vir})$.

We go further and use a characterization of structured holomorphic factorization algebras on $\mathbf{C}$ from the book [9] to show that this factorization algebra has the structure of a vertex algebra and it is equivalent to that of the Virasoro vertex algebra. In [9], a functor $\mathbb{V}\mathrm{ert}$ from the category of structured holomorphic factorization algebras on $\mathbf{C}$ to the category of vertex algebras is defined. The main result from the first part of this paper can be stated as follows.

Theorem 1 *For any complex number $c \in \mathbf{C}$, there is a factorization algebra $\mathcal{V}\mathrm{ir}_c$ on $\mathbf{C}$ (given by the enveloping factorization algebra for the extension of $\mathcal{L}^{\mathbf{C}}$ by the cocycle $c\omega$) which determines a vertex algebra $\mathbb{V}\mathrm{ert}(\mathcal{V}\mathrm{ir}_c)$. Moreover, there is an isomorphism of vertex algebras*

$$\mathbf{Vir}_c \xrightarrow{\cong} \mathbb{V}\mathrm{ert}(\mathcal{V}\mathrm{ir}_c)$$

where $\mathbf{Vir}_c$ denotes the Virasoro vertex algebra of charge c.

After spelling out the local structure, we study global sections, or the factorization homology, of $\mathcal{V}\mathrm{ir}_c$. Some care must be taken when defining the cocycle determining the extension on the Dolbeault resolution of vector fields on a general Riemann surface since the original cocycle for the local Lie algebra on $\mathbf{C}$ is coordinate dependent. We show that a slightly modified version of the cocycle gives a coordinate independent description and hence a *universal* version of the cocycle. That is, we show that the Virasoro factorization algebra defines a factorization algebra on the site of Riemann surfaces. We calculate the cohomology of global sections of the Virasoro factorization algebra and write down correlation functions.

This paper can be viewed in conjunction with a new direction of work that combines methods of renormalization, homological perturbation theory, and factorization algebras developed in Costello [8] and Costello–Gwilliam [9,10]. From the data of a classical field theory, defined in terms of an action functional, one applies of homotopical renormalization to construct a quantization. Locality of the theory and the quantization on the manifold in which the theory lives combine to give the structure of a *factorization algebra* on the algebraic observables of the theory.

The last section of this paper exhibits how the usual physical idea of the Virasoro algebra encoding the symmetries of a conformal field theory fits in to the model for QFT developed by Costello–Gwilliam. The usual Virasoro symmetry in field theory is naturally encoded by map of factorization algebras from the Virasoro factorization algebra (at a certain central charge) to the factorization algebra of observables. We will focus on a particular example of a chiral conformal field theory, called the *free $\beta\gamma$ system*, though the methods we use work in a much larger context.

One motivation for this approach to constructing the Virasoro vertex algebra and realizing its significance in the context of conformal field theory is that it is amenable to higher dimensions. Indeed, it is completely natural to consider field theories defined on a complex manifold that have the symmetry of holomorphic vector fields. This is a generalization of a chiral conformal field theory in complex dimension one and examples are bountiful in the context of supersymmetric theories in higher dimensions [17,18]. On completely formal grounds, we then expect the quantization of such a theory (assuming the relevant anomalies vanish) to have the symmetry of a central extension of this local Lie algebra of holomorphic vector fields. In future work we classify such central extensions and understand them as certain universal characteristic classes on the moduli stack of complex structures. We also investigate the local structure of a higher-dimensional factorization algebra on $\mathbf{C}^d$, for $d > 1$, analogous to the way holomorphic factorization algebras on $\mathbf{C}$ give rise to vertex algebras. Thus, this can be seen as an approach to studying higher-dimensional generalizations of vertex algebras.

1.1 Notation and conventions

- If X is a complex manifold we have a decomposition of the tangent bundle $T^{1,0}X \oplus T^{0,1}X$. Unless otherwise noted we will write $TX = T^{1,0}X$ for the $(1, 0)$ part of the tangent bundle. With respect to this decomposition the de Rham differential

$$d_{\mathrm{dR}} : \mathcal{O}(X) \to \Omega^1(X) = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X) := \Gamma\left((T^{1,0}X)^\vee\right) \oplus \Gamma\left((T^{0,1}X)^\vee\right)$$

splits as $\partial + \bar{\partial}$.

– Let V be a graded vector space. We denote by $\mathrm{Tens}(V)$ the full tensor algebra of V, $\oplus_{n\geq 0} V^{\otimes n}$. This is again a graded vector space in the natural way. Define the symmetric algebra as

$$\mathrm{Sym}(V) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(V)$$

where $\mathrm{Sym}^n(V) = (V \otimes \cdots \otimes V)_{\Sigma_n}$. We will also need the completed symmetric algebra

$$\hat{\mathrm{Sym}}(V) = \prod_{n\geq 0} \mathrm{Sym}^n(V).$$

– All graded vector spaces are cohomologically graded. For $k \in \mathbf{Z}$ we denote by $V[k]$ the graded vector space with graded components:

$$(V[k])^i = V^{i+k}.$$

If W is an ordinary (ungraded) vector space, we will understand it as a graded vector space concentrated in degree zero. For instance, $W[k]$ is concentrated in degree $-k$.

– Let $\mathfrak{g}$ be a dg Lie algebra. That is, a $\mathbf{Z}$-graded vector space together with a differential $d_{\mathfrak{g}} : \mathfrak{g}^\bullet \to \mathfrak{g}^{\bullet+1}$ of degree +1 and a bracket $[-,-]$ that is graded antisymmetric, satisfies the graded Jacobi identity, and for which $d_{\mathfrak{g}}$ is a graded derivation. We define Chevalley–Eilenberg chains for computing Lie algebra homology as

$$C_*(\mathfrak{g}) \ := \ \mathrm{Sym}\,(\mathfrak{g}[1]) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(\mathfrak{g}[1])$$

with differential given by $d = d_{\mathfrak{g}} + d_{\mathrm{CE}}$ where d_{CE} is the usual CE-differential determined by $d_{\mathrm{CE}}(a \wedge b) = [a, b]$ on Sym^2. Similarly, Chevalley–Eilenberg cochains for computing Lie algebra cohomology are defined by

$$C^*(\mathfrak{g}) \ := \ \hat{\mathrm{Sym}}\left(\mathfrak{g}^\vee[-1]\right)$$

with differential given by $d = d_{\mathfrak{g}}^\vee + d_{\mathrm{CE}}^\vee$.

– We will need to consider a topology on the dg vector spaces we work with. Unless otherwise noted our complexes will take values in the category of dg nuclear vector spaces. This is an especially convenient category of topological vector spaces which are locally convex and Hausdorff. For more on their properties see [8] Given two dg nuclear vector spaces we denote by $V \otimes W$ the completed tensor product. This tensor product makes the category of dg nuclear vector spaces a symmetric monoidal category which we denote by $\mathrm{dgNuc}^{\otimes}$.

2 Virasoro as a local Lie algebra on C

In this section, we introduce a local version of the Virasoro Lie algebra on the complex plane. It appears as an extension of the Lie algebra of holomorphic vector fields on **C** given by an explicit cocycle.

2.1 Dolbeault resolution of holomorphic vector fields

Let X be a complex manifold. We study the space of holomorphic sections of the holomorphic $(1, 0)$ tangent bundle $\mathcal{O}^{hol}(TX)$. We can use the decomposition of the tangent bundle above gives us a resolution for this space. Indeed, the $\bar{\partial}$ operator extends to define a complex

$$\Omega^{0,0}(X, TX) \xrightarrow{\bar{\partial}} \Omega^{0,1}(X, TX) \xrightarrow{\bar{\partial}} \Omega^{0,2}(X, TX) \xrightarrow{\bar{\partial}} \cdots .$$

We will be concerned with the case that the complex manifold is a Riemann surface Σ. Indeed the Dolbeault complex above defines the dg Lie algebra $\mathcal{L}^{\Sigma} := \Omega^{0,*}(\Sigma, T\Sigma)$. The differential is $\bar{\partial}$, and the Lie bracket is given by extending the ordinary Lie bracket on $\Omega^{0,0}(\Sigma, T\Sigma)$ to a graded Lie bracket.

We will consider $\mathcal{L}^{\Sigma}$ as a sheaf of cochain complexes that assigns to an open $U \subset \Sigma$ the complex

$$(\Omega^{0,*}(U, TU), \bar{\partial}).$$

Moreover, $\mathcal{L}^{\Sigma}$ is a sheaf of dg Lie algebras. In fact it has even more structure, that of a *local DG Lie algebra*.

The following definition can be found in [9].

Definition 2.1 A *local dg Lie algebra* on a manifold M is the following data:

(1) A graded vector bundle L on M, whose sheaf of smooth sections is denoted $\mathcal{L}$.
(2) A differential operator $d : \mathcal{L} \to \mathcal{L}$ of degree one and square 0.
(3) Antisymmetric multi-differential operators

$$d : \mathcal{L} \to \mathcal{L} \quad , \quad [-, -] : \mathcal{L}^{\otimes 2} \to \mathcal{L}$$

of degree one and zero, respectively, that give $\mathcal{L}$ the structure of a sheaf of dg Lie algebras.

We will often refer to a local Lie algebra simply by its sheaf of sections $\mathcal{L}$. A closely related object that we will consider is the associated *precosheaf* of compactly supported sections $\mathcal{L}_c$. To an open set $U \subset M$, it assigns the graded vector space $\Gamma_c(L, U)$ of compactly supported sections of L supported on U. Since the differential and bracket are multi-differential operators of the bundle L, we see that they restrict to define the structure of a dg Lie algebra on $\mathcal{L}_c(U)$. Note, however, that although $\mathcal{L}_c$ is a cosheaf of underlying graded vector spaces on M, it is *not* a cosheaf of dg Lie algebras (though it is still a prechosheaf of dg Lie algebras). This follows from the fact that the coproduct in the category of dg Lie algebras differs from the direct sum.

Since $\bar{\partial}$ and the Lie bracket of vector fields are differential and bi-differential operators, respectively, we see that $\mathcal{L}^{\Sigma}$ is a local dg Lie algebra that assigns to an open set $U \subset \Sigma$ the dg Lie algebra $\mathcal{L}^{\Sigma}(U) = (\Omega^{0,*}(U, TU), \bar{\partial})$. The precosheaf of compactly supported sections $\mathcal{L}^{\Sigma}_c$ assigns to an open set $U \subset \Sigma$ the dg Lie algebra $\mathcal{L}^{\Sigma}_c(U) = (\Omega^{0,*}_c(U, TU), \bar{\partial})$

2.2 Lie algebra extensions and the cocycle

We are interested in a one-dimensional central extension of $\mathcal{L}^{\Sigma}$. As the Lie algebra in question is local, we ask for our extensions to be local as well. Before defining what we mean by this, we review extensions of ordinary Lie and dg Lie algebras.

In the remainder of this section, as well Sects. 3 and 4 we will be concerned with the case that the Riemann surface is the complex line $\Sigma = \mathbf{C}$.

2.2.1 Extensions

A central extension $\hat{\mathfrak{g}}$ of an ordinary Lie algebra $\mathfrak{g}$ is a Lie algebra that fits into an exact sequence

$$0 \to \mathbf{C} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

such that $[\lambda, x] = 0$ for all $\lambda \in \mathbf{C}$ and $x \in \mathfrak{g}$. Isomorphism classes of central extensions of $\mathfrak{g}$ are in bijective correspondence with $\mathrm{H}^2(\mathfrak{g})$.

For a dg Lie algebra $\mathfrak{g}$ and an integer k, we can define a k-shifted central extension of $\mathfrak{g}$. It fits into and exact sequence

$$0 \to \mathbf{C}[k] \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{1}$$

and satisfies $[\lambda, x] = 0$ as above.

Remark 1 The group $H^{2+k}(\mathfrak{g})$ does *not* parametrize such extensions. It parametrizes a larger class of extensions, namely shifted L_∞-extensions of $\mathfrak{g}$. That is, exact sequences as in (1) except $\hat{\mathfrak{g}}$ is allowed to be an L_∞-algebra, and the maps are of L_∞-algebras.

Example 1 Consider the Lie algebra of vector fields on S^1, $\mathrm{Vect}(S^1)$. This is an ordinary Lie algebra that, as usual, can be thought of as a dg Lie algebra concentrated in degree zero. The Gelfand–Fuks extension mentioned in the introduction is a 2-cocycle, hence determines an element in $H^2_{\mathrm{Lie}}(\mathrm{Vect}(S^1))$. In fact, this cohomology is one-dimensional. See [14] for a proof of this.

Now, let $\mathcal{L}$ be a local dg Lie algebra on a manifold M. A *local k-shifted central extension* of $\mathcal{L}$ is a dg Lie algebra structure on the precosheaf

$$\hat{\mathcal{L}}_c = \mathcal{L}_c \oplus \underline{\mathbf{C}}[k]$$

such that for all opens $U \subset M$:

- (Central) For any $\lambda \in \mathbf{C}[k]$ and $x \in \hat{\mathcal{L}}_c(U)$, we have $[x, \lambda] = 0$ and the sequence

$$0 \to \mathbf{C}[k] \to \hat{\mathcal{L}}_c(U) \to \mathcal{L}_c(U) \to 0$$

  is exact.
- (Local) The differential $d_{\hat{\mathcal{L}}} : \mathcal{L}_c(U) \to \mathbf{C}[k]$ and Lie bracket $[-,-] : \mathcal{L}_c(U) \otimes \mathcal{L}_c(U) \to \mathbf{C}[k]$ both factor through the k-shifted integration map

$$\int_U : \mathrm{Dens}_c(U)[-k] \to \mathbf{C}[k].$$

  Here, Dens_c denotes the cosheaf of compactly supported densities.

Remark 2 As in the ordinary case, there is a cohomology that parametrizes central extensions of this local nature. Let $\mathcal{L}$ be a local Lie algebra on M. In [8] local functionals are defined as

$$C^*_{\mathrm{loc,red}}(\mathcal{L}) := \mathrm{Dens}_M \otimes_{\mathcal{D}_M} C^*_{\mathrm{red}}\left(\mathrm{Jet}(L)\right).$$

Here, $\mathcal{D}_M$ is the space of differential operators on M and $\mathrm{Jet}(L)$ is the infinite Jet-bundle of the vector bundle L. The jet-bundle inherits a natural $\mathcal{D}_M$-module structure, and this induces one on cochains. There is another interpretation of local cochains. They are precisely the graded multilinear functionals on $\mathcal{L}$ that factor through the integration map. More precisely, integration along M induces a natural inclusion

$$C^*_{\mathrm{loc,red}}(\mathcal{L}) \hookrightarrow C^*_{\mathrm{red}}(\mathcal{L}_c(M))$$

that sends a local functional S to the functional $\varphi \mapsto \int_M S(\varphi)$.

Just as in the case of (non-local) dg Lie algebras, the degree $2+k$ cocycles of $C^*_{\mathrm{loc,red}}(\mathcal{L})$ parametrize a larger class of extensions, namely local L_∞-algebra extensions of $\mathcal{L}$. For our situation, when $\mathcal{L}$ is the Dolbeault resolution of holomorphic vector fields on $U \subset \mathbf{C}$ (or on a closed Riemann surface Σ), the non-trivial degree one cocycles are all cohomologous to one of the form

$$\mathcal{L}_c(U)^{\otimes 2} \to \mathbf{C}$$

and hence all (-1)-shifted extensions will be equivalent to a local dg Lie algebra.

2.2.2 A cocycle for $\mathcal{L}^{\mathbf{C}}$

We now define the cocycle used to construct the central extension of $\mathcal{L}^{\mathbf{C}}$ we are interested in. Let $U \subset \mathbf{C}$ and fix a coordinate. Consider the bilinear map

$$\omega : \mathcal{L}^{\mathbf{C}}_c(U) \otimes \mathcal{L}^{\mathbf{C}}_c(U) \to \mathbf{C}[-1]$$

given by

$$(\alpha \otimes \partial_z, \beta \otimes \partial_z) \mapsto \frac{1}{2\pi} \frac{1}{12} \int_U \left(\partial_z^3 \alpha_0 \beta_1 + \partial_z^3 \alpha_1 \beta_0 \right) \mathrm{d}^2 z$$

where $\alpha = \alpha_0 + \alpha_1 \mathrm{d}\bar{z}$ and $\beta = \beta_0 + \beta_1 \mathrm{d}\bar{z}$. The target of the bilinear map ω is $\mathbf{C}[-1]$ which reflects the fact that it is only nonzero when the degree of $\alpha\partial_z$ and $\beta\partial_z$ sum to 1. One checks by direct calculation that ω is a cocycle. It is our analog of the Gelfand–Fuks cocycle.

This cocycle defines for us a local (-1)-shifted central extension $\hat{\mathcal{L}}^{\mathbf{C}}$ of $\mathcal{L}^{\mathbf{C}}$ via the local extension construction above. As a cosheaf of vector spaces, it is

$$\mathcal{L}_c^{\mathbf{C}} \oplus \mathbf{C} \cdot c[-1].$$

On an open U, the Lie bracket is defined by the rules

$$[\alpha \otimes \partial_z, \beta \otimes \partial_z]_{\hat{\mathcal{L}}_c^{\mathbf{C}}} := [\alpha \otimes \partial_z, \beta \otimes \partial_z] + \frac{1}{2\pi} \frac{1}{12} \int_U \left(\partial_z^3 \alpha_0 \beta_1 + (\partial_z^3 \alpha_1 \beta_0 \right) \mathrm{d}^2 z \cdot c$$

and $[\alpha \otimes \partial_z, c]_{\hat{\mathcal{L}}_c^{\mathbf{C}}} = 0$.

The locality and cocycle properties imply that ω determines an element in $H^1_{\mathrm{loc}}(\mathcal{L}^{\mathbf{C}})$.

2.3 Statements about cohomology

The following facts about the $\bar{\partial}$-cohomology of subsets of $\mathbf{C}$ will be used throughout. Let $U \subset \mathbf{C}$ be open. The following lemma is due to Serre [21].

Lemma 1 *Let E be an arbitrary holomorphic vector bundle on a one-dimensional complex manifold U. The compactly supported Dolbeault cohomology with coefficients in E is concentrated in degree 1, and there is a continuous isomorphism*

$$H^1(\Omega_c^{0,*}(U; E), \bar{\partial}) \cong \left(\Omega^1_{\mathrm{hol}}(U; E^\vee) \right)^\vee$$

where $E^\vee$ denotes the linear dual of the bundle E and the outer dual $(-)^\vee$ of holomorphic one-forms with values in $E^\vee$ denotes the continuous linear dual of nuclear Fréchet spaces.

The isomorphism can explicitly be written as follows. We assign to an element $\alpha \in \Omega^{0,1}(U; E)$, the continuous linear functional

$$\langle \alpha, - \rangle : \Omega^1_{\mathrm{hol}}(U) \to \mathbf{C} \ , \ \ \beta \mapsto \int_U \langle \alpha, \beta \rangle_E$$

where $\langle -, - \rangle_E$ denotes the evaluation pairing between E and $E^\vee$.

Next, we need the following fact about dg Lie algebras.

Lemma 2 *Suppose L is a dg Lie algebra such that $H^*(L)$ is concentrated in a single degree. Then L is formal (as a dg Lie algebra).*

Proof Suppose the cohomology of L is concentrated in degree m. Define the subcomplex $L' \hookrightarrow L$ as follows: for $k < m$ set $(L')^k := L^k$, for $k = 0$ set $(L')^0 = \ker(d_L : L^0 \to L^1)$, for $k > m$ set $(L')^k := 0$. There is a natural zigzag of dgla's

$$L \hookleftarrow L' \to H^0 L.$$

Both arrows are clearly weak equivalences.

Serre's result implies that $\mathcal{L}_c^{\mathbf{C}}(U) = \Omega_c^{0,*}(U, TU)$ is formal for all opens $U \subset \mathbf{C}$. In fact, there is a quasi-isomorphism of dg Lie algebras

$$\Omega_c^{0,*}(U, TU) \simeq H(\Omega_c^{0,*}(U, TU), \bar{\partial}) \cong \left(\Omega^1_{\mathrm{hol}}(U, T^*U)\right)^\vee [-1].$$

This implies the following useful fact about the Lie algebra cohomology.

Proposition 2 *Let $U \subset \mathbf{C}$ be open. Then,*

$$H_*^{\mathrm{Lie}}(\mathcal{L}_c^{\mathbf{C}}(U)) := H^*\left(\mathrm{Sym}(\mathcal{L}_c^{\mathbf{C}}(U)[1]), \bar{\partial} + d_{CE}\right) \cong \mathrm{Sym}\left(\Omega^1_{\mathrm{hol}}(U, T^*U)^\vee\right)$$

concentrated in degree 0.

Here, we extend the differential $\bar{\partial}$ on $\mathcal{L}_c(U)$ to the symmetric algebra in the obvious way.

Proof This result follows from formality. Indeed,

$$H_*^{\mathrm{Lie}}(\mathcal{L}_c^{\mathbf{C}}(U)) \cong H_*^{\mathrm{Lie}}(\mathrm{H}^*_{\bar{\partial}}(\mathcal{L}_c^{\mathbf{C}}(U)) = H_*^{\mathrm{Lie}}(\Omega^1_{\mathrm{hol}}(U, T^*U)^\vee[-1]).$$

Now, $\Omega^1_{\mathrm{hol}}(U, T^*U)^\vee[-1]$ is an abelian dg Lie algebra concentrated in a single degree. Thus

$$H_*^{\mathrm{Lie}}(\Omega^1_{\mathrm{hol}}(U, T^*U)^\vee[-1]) = \mathrm{Sym}\left(\Omega^1_{\mathrm{hol}}(U, T^*U)^\vee\right)$$

as desired.

This result also follows from considering the filtration spectral associated to symmetric tensor power degree. The E_1-page is $\mathrm{Sym}\left(H^*_{\bar{\partial}}(\mathcal{L}_c^{\mathbf{C}}(U))\right)$ and the spectral sequence degenerates at the E_2-page as the $\bar{\partial}$-cohomology is concentrated in a single degree. In fact, the degeneration of this spectral sequence associated to cochains on a dg Lie algebra $\mathfrak{g}$ is closely related to the formality of $\mathfrak{g}$, for example see [19].

Remark 3 In the second part of the paper, we consider closed Riemann surfaces. It is still true that on a closed Riemann surface, the spectral sequence associated to the dg Lie algebra $\Omega^{0,*}(\Sigma, T\Sigma)$ degenerates. In fact, this dg Lie algebra is also formal.

2.4 Factorization algebras

Central to this work is the notion of a *factorization algebra*. We recall the relevant theory as in [9].

Fix a topological space M. For the level of generality of most of this section, we work in an arbitrary symmetric monoidal category $\mathcal{C}^{\otimes}$ closed under small colimits. For the purposes of this work, we are mainly concerned with $\mathcal{C} = \mathrm{dgNuc}$, the dg category of cochain complexes of nuclear vector spaces over $\mathbf{C}$ with symmetric monoidal structure given by the completed tensor product over $\mathbf{C}$.

2.4.1 Prefactorization

A *prefactorization algebra* $\mathcal{F}$ on M with values in $\mathcal{C}^{\otimes}$ is an assignment of an object $\mathcal{F}(U)$ of $\mathcal{C}$ for each open $U \subset M$ together with the following data:

- For $U \subset V$, a morphism $\mathcal{F}(U) \to \mathcal{F}(V)$.
- For any finite collection $\{U_i\}$ of pairwise disjoint opens in an open $V \subset M$ a morphism

$$\otimes_i \mathcal{F}(U_i) \to \mathcal{F}(V).$$

- Coherences between the above two sets of data.

For a better definition, we need to define the following symmetric monoidal category $\mathrm{Fact}(M)^{\sqcup}$. Its objects are topological spaces U together with a map $U \to M$ such that on each connected component of U this map is an open embedding. A morphism from $U \to M$ to $V \to M$ is a commutative diagram

$$\begin{tikzcd} U \arrow[rr, "i"] \arrow[dr] & & V \arrow[dl] \\ & M & \end{tikzcd}$$

with i an open embedding. Composition is done in the obvious way. The symmetric monoidal structure is given by disjoint union.

A more precise definition of a *prefactorization algebra* is symmetric monoidal functor

$$\mathcal{F} : \mathrm{Fact}(M)^{\sqcup} \to \mathcal{C}^{\otimes}.$$

Example 2 The coherence of the data above can be read of immediately from this definition and encodes the transitivity of opens. For instance, suppose $U_1, U_2 \subset V \subset W$ are opens with U_i disjoint. Then $\mathcal{F}$ applied to this composition says that

$$\begin{tikzcd} \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \arrow[r] \arrow[dr] & \mathcal{F}(V) \arrow[d] \\ & \mathcal{F}(W) \end{tikzcd}$$

commutes.

The structures we consider in the first part of this paper are completely encoded by a *pre*factorization structure. In the last section, however, when we will be concerned with global sections on a general Riemann surface, it is critical that our object satisfies a form of descent.

2.4.2 Factorization: gluing

A *factorization algebra* is a prefactorization algebra satisfying a descent axiom. Descent for ordinary sheaves (or cosheaves) says that one can recover the value of the sheaf on large open sets by breaking it up into smaller opens. That is, if $\mathcal{U} = \{U_i\}$ is a cover of $U \subset M$, then a presheaf $\mathcal{F}$ of vector spaces is a sheaf iff

$$\mathcal{F}(U) \to \bigoplus_i \mathcal{F}(U_i) \rightrightarrows \oplus_{i,j} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram for all opens U and covers $\mathcal{U}$. It is convenient to introduce the Čech complex associated to $\mathcal{U}$. The pth space is

$$\check{C}^p(\mathcal{U}, \mathcal{F}) := \bigoplus_{i_0,\ldots,i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}).$$

The differential $\check{C}^p \to \check{C}^{p+1}$ is induced from the natural inclusion maps $U_{i_0} \cap \cdots \cap U_{i_p} \hookrightarrow U_{i_0} \cap \cdots \cap \hat{U}_{i_j} \cap \cdots \cap U_{i_p}$. The sheaf condition is equivalent to saying that the natural map

$$\mathcal{F}(U) \to H^0(\check{C}(\mathcal{U}, \mathcal{F}))$$

is an isomorphism. There is a similar construction for cosheaves, but the arrow goes in the opposite direction.

We are interested in descent for a different topology, that is, for only a special class of open covers. Call an open cover $\mathcal{U} = \{U_i\}$ of $U \subset M$ a *Weiss cover* if for any finite collection of points $\{x_1, \ldots, x_k\}$ in U, there exists an open set U_i such that $\{x_1, \ldots, x_k\} \subset U_i$. This is equivalent to providing a topology on the Ran space.

A Weiss cover defines a Grothendieck topology on $\mathrm{Op}(M)$, the poset of opens in M. A *factorization algebra* on M is a prefactorization algebra on M that is, in addition, a homotopy cosheaf for this Weiss topology.

When $\mathcal{C}^{\otimes} = \mathrm{dgVect}$, we can be explicit about this homotopy gluing condition using a variant of the Čech complex above. Let $\mathcal{F}$ be a cosheaf of dg vector spaces. For $\mathcal{U} = \{U_i\}_{i \in I}$ let $\check{C}^p(\mathcal{U}, \mathcal{F})$ be the complex

$$\bigoplus_{i_0,\ldots,i_p} \mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_k})[p-1]$$

with differential inherit from $\mathcal{F}$. Then $\check{C}(\mathcal{U}, \mathcal{F})$ is a bigraded object. The differential is the total differential obtained from combining the ordinary Čech differentials above plus the internal differential of $\mathcal{F}$. The cosheaf condition is that the natural map

$$\check{C}(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is an equivalence for all Weiss covers $\mathcal{U}$ of U.

Remark 4 One might refer to this as a *homotopy* factorization algebra, reserving a *strict* factorization algebra for one in which

$$\check{H}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(U)$$

is an equivalence. The $\check{H}^0$ means we have only taken cohomology with respect to the Čech differentials. It has a natural dg structure inherited from $\mathcal{F}$.

2.5 (Twisted) envelopes

One of the most useful ways of constructing factorization algebras is the "factorization envelope" of a local Lie algebra. This is the analog of the universal enveloping algebra of a Lie algebra.

Let $\mathcal{L}$ be any local Lie algebra on a manifold M. Denote by $\mathcal{L}_c$ its associated cosheaf of compactly supported sections. Define the prefactorization algebra $U^{\text{fact}}\mathcal{L}$ as follows:

- For an open $U \subset M$, we assign the complex $C_*(\mathcal{L}_c(U))$ with it's usual differential $d = d_{\mathcal{L}} + d_{\text{CE}}$.
- Suppose $\sqcup_i U_i \hookrightarrow V$ is an inclusion of disjoint opens inside a bigger open. The structure maps of the prefactorization algebra come from applying $C_*(-)$ to the structure maps of the cosheaf

$$\oplus_i \mathcal{L}_c(U_i) \to \mathcal{L}_c(V).$$

In fact, we will use the following fact to compute global sections, i.e., factorization homology.

Theorem 3 ([9]) *The prefactorization algebra $U^{\text{fact}}\mathcal{L}$ satisfies descent, that is it is a factorization algebra.*

Example 3 If $\mathfrak{g}$ is an *ordinary* Lie algebra, we can consider the local Lie algebra $\Omega^*_{\mathbf{R}} \otimes \mathfrak{g}$ on $\mathbf{R}$. The factorization algebra $U^{\text{fact}}(\Omega_{\mathbf{R}} \otimes \mathfrak{g})$ is locally constant on $\mathbf{R}$. Now, map that sends a factorization algebra on $\mathbf{R}$ to its value on an interval is known to induce an equivalence of categories

$$\{A_\infty\text{−algebras}\} \simeq \{E_1\text{−algebras}\} \simeq \{\text{ locally constant factorization algebras on } \mathbf{R}\}.$$

Under this equivalence the factorization algebra $U^{\text{fact}}\mathfrak{g}$ corresponds to the associative algebra $U\mathfrak{g}$.

Now, suppose we have an element $\omega \in H^1_{\text{loc}}(\mathcal{L})$ corresponding to a (-1)-shifted central extension $\hat{\mathcal{L}}$ of a local Lie algebra $\mathcal{L}$ on a manifold M. We define the *twisted factorization envelope* $U^{\text{fact}}_{\omega}\mathcal{L}$ as the factorization algebra on M that sends an open $U \subset M$ to the complex

$$(\text{Sym}\,(\mathcal{L}(U)[1] \oplus \mathbf{C} \cdot C)\,,\, d_{\mathcal{L}} + d_{\text{CE}} + \omega)$$

where ω is made into an operator on Sym as follows. On $\text{Sym}^{\leq 1}$ it is zero, and on Sym^2 it is

$$(\alpha, \beta) \mapsto C \cdot \omega(\alpha, \beta).$$

It is extended to the full symmetric algebra by demanding that it is a graded derivation. Note that $U^{\text{fact}}_{\omega}\mathcal{L}^{\mathbf{C}} = U^{\text{fact}}\hat{\mathcal{L}}^{\mathbf{C}}_c$ so that the twisted envelope is just the envelope of the extended local Lie algebra.

2.5.1 The Virasoro factorization algebra

We will now specialize to factorization algebras valued in the symmetric monoidal category of dg nuclear vector spaces dgNuc or slight variants thereof.

In the remainder of the paper, we are interested in both the untwisted and twisted factorization envelopes of the local Lie algebra of holomorphic vector fields.

First, define the *Virasoro factorization algebra at central charge zero* by

$$\mathcal{V}\text{ir}_0 := U^{\text{fact}}\mathcal{L}^{\mathbf{C}}.$$

This is a factorization algebra valued in the category dgNuc (since the Dolbeault complexes belong to this category).

Let $\omega \in H^1_{\text{loc}}(\mathcal{L}^{\mathbf{C}})$ denote the cocycle from Sect. 2.2.2. We define the *Virasoro factorization algebra* by

$$\mathcal{V}\text{ir} := U^{\text{fact}}_{\omega}\mathcal{L}^{\mathbf{C}}.$$

The factorization algebra $\mathcal{V}\text{ir}$ is a factorization algebra in the category of $\mathbf{C}[c]$-modules in dg nuclear vector spaces. In particular, we can specialize a value of c to obtain a factorization algebra in dgNuc. We will denote such a specialization by $\mathcal{V}\text{ir}_c$ and call it the *Virasoro factorization algebra of central charge* c.

3 Annuli: recovering the Virasoro

In this section, we show how the Virasoro Lie algebra is encoded in the factorization algebras constructed above.

First we recall the definition the Virasoro Lie algebra. Consider the ring of Laurent power series in one variable $\mathbf{C}((t))$. As a vector space the Lie algebra of derivations $W_1^{\times} := \text{Der}\,(\mathbf{C}((t)))$ is isomorphic to $\mathbf{C}((t))\partial_t$. The ring $\mathbf{C}((t))$ is equal to functions on the holomorphic formal punctured disk $\hat{D}^\times$ and $\mathrm{W}_1^\times$ is the Lie algebra of formal vector fields on the punctured disk. Let Vir be the central extension of $\mathrm{W}_1^\times$ determined by the Gelfand–Fuks cocycle ω_{GF} defined in the introduction. It fits into the exact sequence of Lie algebras

$$0 \to \mathbf{C} \cdot C \to \mathrm{Vir} \to W_1^\times \to 0.$$

Thus, as a vector space we have $\mathrm{Vir} = \mathbf{C}((t))\partial_t \oplus \mathbf{C} \cdot C$. Explicitly, the bracket in this Lie algebra is

$$[f(t)\partial_t, g(t)\partial_t] = (f(t)g'(t) - f'(t)g(t))\partial_t + \frac{1}{12}\oint f'''(t)g(t)\mathrm{d}t \cdot C.$$

It is topologically generated by c and $L_n = t^{n+1}\partial_t$ and in terms of these generators, the commutator is

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{m^3 - m}{12}\delta_{n,-m} \cdot C.$$

Now, consider the universal enveloping algebra of the Virasoro Lie algebra $U(\mathrm{Vir})$. Being an associative algebra, it determines a locally constant factorization algebra on $\mathbf{R}_{>0}$. Denote this factorization algebra by $\mathcal{A}_{\mathrm{Vir}}$. Explicitly, $\mathcal{A}_{\mathrm{Vir}}$ sends an interval I to $U(\mathrm{Vir})$ (considered as a dg vector space concentrated in degree zero) and the structure maps are induced by the usual associative multiplication on $U(\mathrm{Vir})$.

Let $\rho : \mathbf{C}^\times \to \mathbf{R}_{>0}$ be the map $z \mapsto z\bar{z}$. We consider the push-forward factorization algebra $\rho_* \mathcal{V}\mathrm{ir}$. This is a factorization algebra on $\mathbf{R}_{>0}$. The main result of this section can be stated as follows.

Proposition 4 *There is a map of factorization algebras*

$$\Phi : \mathcal{A}_{\mathrm{Vir}} \to H^0(\rho_* \mathcal{V}\mathrm{ir}) \tag{2}$$

that is a dense inclusion of topological vector spaces on every open interval $I \subset \mathbf{R}_{>0}$.

Note that on an open interval $I \subset \mathbf{R}_{>0}$

$$(\rho_* \mathcal{F}_\omega)(I) \;=\; \mathcal{V}\mathrm{ir}(\rho^{-1}(I)).$$

So, we need to understand what $\mathcal{V}$ir does to annuli.

Remark 5 This proposition says that every cohomology class in $\mathcal{V}$ir applied to an annulus is arbitrarily close to some element of the universal enveloping algebra of the Virasoro Lie algebra. Moreover, the structure maps of the factorization algebra are the continuous extensions of the multiplication for $U\mathrm{Vir}$.

3.1 The case of zero central charge

Recall that we have the following identification for any open $U \subset \mathbf{C}$:

$$H^*(\mathcal{V}\mathrm{ir}_0(U)) \cong \mathrm{Sym}\left(\mathrm{H}^1(\Omega_c^{0,*}(U, TU))\right) \cong \mathrm{Sym}\left(\Omega^1_{\mathrm{hol}}(U, TU)^\vee\right)$$

concentrated in cohomological degree 0.

First, we describe the untwisted version of the map (2), denote it $\Phi_{un} : \mathbf{C}((z))\partial_z \to \rho_*(\mathcal{V}\mathrm{ir}_0)$. Let $L_n = z^{n+1}\partial \in \mathbf{C}((z))\partial_z$ be the usual basis vectors for $n \in \mathbf{Z}$. Pick an open interval $I \subset \mathbf{R}_{>0}$ and let $A = \rho^{-1}(I)$. We will utilize a function $f : \mathbf{C}^\times \to \mathbf{R}$ for A that satisfies the following:

- f is only a function of $r^2 = z\bar{z}$.
- $\int_A f \, \mathrm{d}z\mathrm{d}\bar{z} = 1$.
- $f(r^2) \geq 0$ and f is supported on A.

We will refer to f as a *bump function* for A. Finally, we define

$$\Phi_{un}(I) : L_n \mapsto \left\lfloor f(z\bar{z})z^{n+2}\mathrm{d}\bar{z}\partial_z \right\rfloor$$

where $\lfloor - \rfloor$ denotes the cohomology class in compactly supported Dolbeault forms. Note that this map is a dense inclusion of topological vector spaces by Serre's resulted stated above. Therefore, we might unambiguously confuse L_n with its image in $H^*(\mathcal{V}\mathrm{ir}_0(A))$. Also, it will be convenient to use the notation $L_n(A) = f(z\bar{z})z^{n+2}\mathrm{d}\bar{z}\partial_z$ for the lift of L_n to the factorization algebra. We make no reference to the bump function chosen since this choice will not affect the cohomology class.

Consider three nested disjoint annuli A_1, A_2, A_3 where A_i has inner radius r_i and outer radius R_i so that $R_1 < r_2$ and $R_2 < r_3$. Suppose all three are contained in the big annuli A, i.e., $r < r_1$ and $R_3 < R$.

Let's explain some notation for the factorization product of such nested annuli. The relevant factorization maps are

$$\begin{aligned} \bullet &: \mathcal{V}\mathrm{ir}_0(A_2) \otimes \mathcal{V}\mathrm{ir}_0(A_1) \to \mathcal{V}\mathrm{ir}_0(A) \\ \bullet &: \mathcal{V}\mathrm{ir}_0(A_3) \otimes \mathcal{V}\mathrm{ir}_0(A_2) \to \mathcal{V}\mathrm{ir}_0(A). \end{aligned}$$

Moving outward, radially, corresponds to multiplying from the right to left in this notation. This is known as *radial ordering*. Using this notation, upon taking cohomology we want to show

$$L_m \bullet L_n - L_n \bullet L_m = (m-n)L_{n+m}.$$

Remark 6 This is a bit of abuse of notation, as we are using the same symbol L_m even though the two live in different spaces. This is a superficial confusion since Φ_{un} is an embedding, but what the above expression actually means is

$$\begin{aligned} &\Phi_{un}(\rho(A_2))(L_m) \bullet \Phi_{un}(\rho(A_1))(L_n) - \Phi_{un}(\rho(A_3))(L_n) \bullet \Phi_{un}(\rho(A_2))(L_m) \\ &\quad = (m-n)\Phi_{un}(\rho(A))(L_{n+m}). \end{aligned}$$

Let $f_i : \mathbf{C}^\times \to \mathbf{R}$ be a bump function for A_i, $i = 1, 2, 3$. We use these to obtain lifts of L_n's to the factorization algebra. Explicitly, $L_m(A_1) \in \mathcal{V}\mathrm{ir}_0(A_1)$, $L_m(A_3) \in \mathcal{V}\mathrm{ir}_0(A_3)$ and $L_n(A_2) \in \mathcal{V}\mathrm{ir}_0(A_2)$.

Now, in cohomology

$$\lfloor L_m(A_1)L_n(A_2) - L_n(A_2)L_m(A_3)\rfloor = L_m \bullet L_n - L_n \bullet L_m$$

and

$$(m-n)L_{m+n} = \lfloor [L_m, L_n](A)\rfloor = \left\lfloor f_2(r^2)(m-n)z^{n+m+2}\mathrm{d}\bar{z} \otimes \partial_z \right\rfloor.$$

Consider the function

$$F(z,\bar{z}) = z^{m+1}\int_0^{z\bar{z}} f_1(s) - f_3(s)\,\mathrm{d}s.$$

We compute the $\bar{\partial}$ operator acting on $F(z,\bar{z})$ as

$$\begin{aligned}\bar{\partial}(F(z,\bar{z})) &= z^{m+1}\frac{\partial}{\partial\bar{z}}\left(\int_0^{z\bar{z}} f_1(s) - f_3(s)\,\mathrm{d}s\right)\mathrm{d}\bar{z}\\ &= z^{m+1}\frac{\partial(z\bar{z})}{\partial\bar{z}}\frac{\partial}{\partial(z\bar{z})}\left(\int_0^{z\bar{z}} f_1(s) - f_3(s)\,\mathrm{d}s\right)\mathrm{d}\bar{z}\\ &= z^{m+2}\left(f_1(z\bar{z}) - f_3(z\bar{z})\right)\mathrm{d}\bar{z}.\end{aligned}$$

Similarly, we have the element $F(z,\bar{z})\partial_z \in \Omega^{0,*}(A, TA)$ and the formula above implies

$$\bar{\partial}(F(z,\bar{z})\partial_z) = L_m(A_1) - L_m(A_3).$$

Let d denote the differential in $C_*(\mathcal{L}_{\mathbf{C}}(A))$. The above implies

$$d(F(z,\bar{z})\partial_z \cdot L_n(A_2)) = (L_m(A_1) - L_m(A_3))L_n(A_2) + \lfloor F(z,\bar{z})\partial_z, L_n(A_2)\rfloor_{\mathcal{V}\mathrm{ir}_0(A_2)}.$$

We compute

$$\begin{aligned}\lfloor F(z,\bar{z})\partial_z, L_n(A_2)\rfloor_{\mathcal{V}\mathrm{ir}_0(A_2)} &= f_2(r^2)\mathrm{d}\bar{z}\left\lfloor z^{m+1}\partial_z, z^{n+2}\partial_z\right\rfloor + z^{m+n+3}\frac{\partial f_2(r^2)}{\partial z}\mathrm{d}\bar{z}\partial_z\\ &= (m-n-1)L_{m+n}(A_2) + z^{m+n+3}\frac{\partial f_2(r^2)}{\partial z}\mathrm{d}\bar{z}\partial_z.\end{aligned}$$

Combining, obtain

$$L_m \bullet L_n - L_n \bullet L_m - [L_m, L_n] - L_{m+n} + \left\lfloor z^{m+n+3}\frac{\partial f_2(r^2)}{\partial z}\mathrm{d}\bar{z}\partial_z\right\rfloor = 0 \qquad (3)$$

where the bracket denotes the cohomology class. We consider the last term. Introduce the element $z^{n+m+2}\bar{z} f_2(r^2)\partial_z$. Applying the $\bar{\partial}$-operator, we get

$$\begin{aligned}\bar{\partial}(z^{n+m+2}\bar{z} f_2(r^2)\partial_z) &= z^{n+m+2} f_2(r^2)\mathrm{d}\bar{z}\partial_z + z^{n+m+2}\bar{z}\left(\frac{\partial f_2(r^2)}{\partial \bar{z}}\right)\mathrm{d}\bar{z}\partial_z \\ &= L_{n+m}(A_2) + z^{n+m+3}\frac{\partial f(r^2)}{\partial z}\mathrm{d}\bar{z}\partial_z\end{aligned}$$

where in the last line we use the fact that $\frac{\partial}{\partial \bar{z}} f_2(r^2) = z f_2'(r^2)$ and $\frac{\partial}{\partial z r} f_2(r^2) = \bar{z} f_2'(r^2)$. Thus, in cohomology we have

$$\left\lfloor z^{n+m+3}\frac{\partial f_2}{\partial z}\mathrm{d}\bar{z}\partial_z \right\rfloor = L_{n+m}$$

so that (3) simplifies to

$$L_m \bullet L_n - L_n \bullet L_m - [L_m, L_n] = 0.$$

3.2 The case of nonzero central charge

We now describe the twisted case. As a vector space, we have

$$\mathrm{Vir} = \mathbf{C}((z))\partial_z \oplus \mathbf{C}\cdot C$$

where c is the central parameter. We recall that the Lie bracket is

$$[L_n, L_m] = (m-n)L_{n+m} + \frac{m^3-m}{12}\delta_{n,-m}c.$$

Again, let $I \subset \mathbf{R}_{>0}$ and write $A = \rho^{-1}(I)$. The map Φ is defined by $\Phi(I)|_{\mathbf{C}((z))\partial_z} = \Phi_{\mathrm{un}}$, and it sends the central parameter of Vir to the central parameter of $\mathcal{L}_c^{\mathbf{C}}(A) \oplus \mathbf{C}\cdot C[-1]$.

The factorization algebra $\mathcal{V}\mathrm{ir}$ assigns to the annulus A the dg vector space:

$$\mathcal{V}\mathrm{ir}(A) = \left(\mathrm{Sym}(\Omega^{0,*}(A, TA)[1] \oplus \mathbf{C}\cdot C), \bar{\partial} + d_{\mathrm{CE}}\right).$$

where $\omega \in C^1(\mathcal{L}^{\mathbf{C}})$ is the central extension as above. We need to show

$$L_m \bullet L_n - L_n \bullet L_m = (m-n)L_{n+m} + \frac{m^3-m}{12}\cdot c.$$

Let the notation be as above. We have

$$\begin{aligned} d(F(z,\bar{z})\partial_z \cdot L_n(A_2)) &= (L_m(A_1) - L_m(A_3))L_n(A_2) + \left[F(z,\bar{z})\partial_z, L_n(A_2)\right]_{\hat{\mathcal{L}}(A_2)} \\ &= (L_m(A_1) - L_m(A_3))L_n(A_2) - (m-n-1)L_{m+n}(A_2) \\ &\quad + z^{m+n+3}\frac{\partial f_2}{\partial z}\mathrm{d}\bar{z}\partial_z - \frac{1}{2\pi}\frac{c}{12}\int_A F(z,\bar{z})\frac{\partial^3}{\partial z^3}\left(f_2(r^2)z^{n+2}\right)\mathrm{d}z\mathrm{d}\bar{z}. \end{aligned}$$

Everything is the same as the zero central charge calculation except for the last line. Applying the same trick as in the previous section to the second line, we see that $d(F(z,\bar{z})\partial_z \cdot L_n(A_2))$ is cohomologous to

$$\begin{aligned} &(L_m(A_1) - L_m(A_3))L_n(A_2) - (m-n)L_{m+n}(A_2) \\ &\qquad - \frac{1}{2\pi}\frac{c}{12}\int_A \frac{\partial^3}{\partial z^3}\left(F(z,\bar{z})\right) f_2(r^2)z^{n+2}\mathrm{d}z\mathrm{d}z \end{aligned}$$

We compute

$$\begin{aligned} \int_A \frac{\partial^3}{\partial z^3}\left(F(z,\bar{z})\right) f_2(r^2)z^{n+2}\mathrm{d}z\mathrm{d}\bar{z} &= \int_A f_2(r^2)z^{n+2}\partial_z^3\left(z^{m+1}\int_0^{z\bar{z}} f_1(s) - f_3(s)\,\mathrm{d}s\right)\mathrm{d}z\mathrm{d}\bar{z} \\ &= (m^3-m)\int_A f_2(r^2)z^{n+m}\mathrm{d}z\mathrm{d}\bar{z} \\ &= (m^3-m)\left(\int_0^{2\pi} e^{i(n+m)\theta}d\theta\right)\left(\int_0^r f_2(r^2)r^{n+m}r\mathrm{d}r\right) \\ &= 2\pi(m^3-m)\delta_{n,-m}. \end{aligned}$$

In the second line, we used the fact that the function $z \mapsto \int_0^{z\bar{z}} f_1 - f_3$ is constant on A_2. Thus, $d(F(z,\bar{z})\partial_z \cdot L_n(A_2))$ is cohomologous to

$$(L_m(A_1) - L_m(A_3))L_n(A_2) - (m-n)L_{m+n}(A_2) - \frac{m^3-m}{12}\delta_{n,-m}\cdot c.$$

Wrapping everything up, in cohomology we have verified

$$0 = L_m \bullet L_n - L_n \bullet L_m - [L_m, L_n] + \frac{m^3-m}{12}\delta_{n,-m}\cdot c$$

as desired.

This completes the proof of Proposition 4.

4 The vertex algebra structure

We sketch the main points of Costello–Gwilliam's treatment of extracting vertex algebras from structured factorization algebras on **C**. We then use their characterization to show that the factorization algebra $\mathcal{V}\mathrm{ir}$ determines a vertex algebra and go further to identify it with the usual Virasoro vertex algebra using the construction.

First, we need to review the definition of a vertex algebra. It consists of a vector space V over the field $\mathbf{C}$ along with the following data:

- A vacuum vector $|0\rangle \in V$.
- A linear map $T : V \to V$ (the translation operator).
- A linear map $Y(-, z) : V \to \mathrm{End}(V)[[z^{\pm 1}]]$ (the vertex operator). We write $Y(v, z) = \sum_{n\in\mathbf{Z}} A_n^v z^{-n}$ where $A_n^v \in \mathrm{End}(V)$.

satisfying the following axioms:

- For all $v, v' \in V$ there exists an $N \gg 0$ such that $A_n^v v' = 0$ for all $n > N$. (This says that $Y(v, z)$ is a *field* for all v).
- (vacuum axiom) $Y(|0\rangle, z) = \mathrm{id}_V$ and $Y(v, z)|0\rangle \in v + zV[[z]]$ for all $v \in V$.
- (translation) $[T, Y(v, z)] = \partial_z Y(v, z)$ for all $v \in V$. Moreover T kills the vacuum.
- (locality) For all $v, v' \in V$, there exists $N \gg 0$ such that

$$(z - w)^N [Y(v, z), Y(v', w)] = 0$$

  in $\mathrm{End}(V)[[z^{\pm 1}, w^{\pm 1}]]$.

We will utilize a reconstruction theorem for vertex algebras. It says that a vertex algebra is completely and uniquely determined by a countable set of vectors, together with a set of fields of the same cardinality and a translation operator subject to a list of axioms.

Theorem 5 (Theorem 2.3.11 of [13]) *Let V be a complex vector space equipped with: an element $|0\rangle \in V$, a linear map $T : V \to V$, a countable set of vectors $\{a^s\}_{s\in S} \subset V$, and fields $A^s(z) = \sum_{n\in\mathbf{Z}} A_n^s z^{-n-1}$ for each $s \in S$ such that:*

- *For all $s \in S$, $A^s(z)|0\rangle \in a^s + zV[[z]]$;*
- *$T|0\rangle = 0$ and $[T, A^s(z)] = \partial_z A^s(z)$;*
- *$A^s(z)$ are mutually local;*
- *and V is spanned by $\{A_{j_1}^{s_1} \cdots A_{j_m}^{s_m}|0\rangle\}$ as the j_i's range over negative integers.*

Then, the data $(V, |0\rangle, T, Y)$ defines a unique vertex algebra satisfying

$$Y(a^s, z) = A^s(z).$$

The main result of this section identifies two vertex algebras: the first comes from the factorization algebra, the other one is the Virasoro vacuum vertex algebra defined in the next section. We prove these are the same using the above reconstruction theorem.

4.1 The Virasoro vertex algebra

We recall the definition of the Virasoro vertex algebra. For us, it will be a vertex algebra over the polynomial ring $\mathbf{C}_c := \mathbf{C}[c]$. For an arbitrary value of c, this will specialize to the usual Virasoro vertex algebra associated to that central charge. First, consider, as we did above, the associative algebra given by the universal envelope of the Virasoro Lie algebra $U = U(\mathrm{Vir})$. There is a subalgebra $U_+ \subset U(\mathrm{Vir})$ generated by elements of the form $z^{n+1}\partial_z$ with $n \geq -1$. Next, define

$$\mathbf{Vir} = \mathrm{Ind}_{U_+}^{U} \mathbf{C}_c = U \otimes_{U_+} \mathbf{C}_c$$

where the L_n's act trivially on $\mathbf{C}_c$ and the central parameter C acts by multiplication by c. The vacuum vector is the natural image of the element $1 \otimes 1 \in U \otimes \mathbf{C}$ in $\mathbf{Vir}_c$. The fields are

$$L(z) := \sum_{n \in \mathbf{Z}} L_n z^{-n-2}$$

and translation operator is $T = L_{-1} = \partial_z$. These satisfy the axioms in the reconstruction theorem, and so define a vertex algebra, simply denoted **Vir**. We will call this $\mathbf{C}[c]$-linear vertex algebra the *Virasoro vertex algebra*. Note that when we specialize to a particular complex number, we obtain the $\mathbf{C}$-linear vertex algebra $\mathbf{Vir}|_{c=c_0} = \mathbf{Vir}_{c_0}$ called the *Virasoro vertex algebra of central charge* c.

4.2 From factorization to vertex

In the first part of this note, we studied a particular two-dimensional factorization algebra and did not mention a vertex algebra. This section is a bit of an aside and sketches the relationship between certain structured factorization algebras on $\mathbf{C}$ and vertex algebras. This relationship is made more precise in [9], but we try to sketch the main points. The main result is essentially a functor from a subcategory of factorization algebras on $\mathbf{C}$ to vertex algebras, and we will use this result to read off the vertex algebra structure from the factorization algebra $\mathcal{V}\mathrm{ir}$ above.

The maps $Y(-, z)$ encode the "multiplication" of the vertex algebra. We can view it has a multiplication parametrized by a complex coordinate $z \in \mathbf{C}$. Consider the two points $0, z \in \mathbf{C}$ with $z \neq 0$. This multiplication has the form

$$Y_z : V_0 \otimes V_z \to V((z))$$

Critical to the structure of a vertex algebra is holomorphicity. Indeed, the axioms imply that the Y_z's vary holomorphically. Thus, the factorization algebra we start with must be translation invariant (so the vector space assigned does not depend on the open set up to translations) together with a holomorphicity condition.

For the remainder of this section, let $\mathcal{F}$ be a prefactorization algebra on $\mathbf{C}$ in the appropriate category of differentiable vector spaces. [1]

We say that $\mathcal{F}$ is *holomorphically translation invariant* if

- $\mathcal{F}$ is translation invariant.

[1] Some care is needed to define this category correctly. We refer the interested reader to [9]

– There exists a degree -1 derivation $\eta : \mathcal{F} \to \mathcal{F}$ such that $\mathrm{d}\eta = \partial_{\bar{z}}$ as derivations of $\mathcal{F}$.

Also important will be the notion of a *smooth S^1-equivariant* structure on $\mathcal{F}$. We will mention this shortly. For now, we discuss how to read off the structure of a vertex algebra from a holomorphic translation-invariant factorization algebra. The key is that such factorization algebra defines a coalgebra structure over a certain (colored) cooperad.

Define the complex manifold

$$\mathrm{Disks}(r_1, \ldots, r_k) := \{z_1, \ldots, z_k \in \mathbf{C} \mid D(z_1, r_1) \sqcup \cdots \sqcup D(z_k, r_k) \text{ disjoint}\} \subset \mathbf{C}^k.$$

The collection of these spaces form a $\mathbf{R}_{>0}$-colored operad in the category of complex manifolds, which we denote Disks. Applying the functor $\Omega^{0,*}$ we get a $\mathbf{R}_{>0}$-colored cooperad $\Omega^{0,*}(\mathrm{Disks})$ in the category of differentiable vector spaces. The main technical fact that we use to read off the structure of a vertex algebra is

Proposition 6 ([9]) *Let $\mathcal{F}$ be a holomorphically translation-invariant factorization algebra on $\mathbf{C}$. Then, $\mathcal{F}$ defines an algebra over the $\mathbf{R}_{>0}$-colored cooperad $\Omega^{0,*}$*(Disks).

This means that at the level of cohomology as we let $p \in \mathrm{Disks}(r_1, \ldots, r_k)$ vary the factorization maps

$$m[p] : H^*\mathcal{F}(D(0, r_1)) \times \cdots \times H^*\mathcal{F}(D(0, r_k)) \to H^*\mathcal{F}(\mathbf{C})$$

lift to a map

$$\mu^{r_1,\ldots,r_k}_{z_1,\ldots,z_k} : H^*\mathcal{F}(D(z_1, r_1)) \times \cdots \times \mathrm{H}^*\mathcal{F}(D(z_k, r_k)) \to \mathrm{Hol}\left(\mathrm{Disks}(r_1, \ldots, r_k), H^*\mathcal{F}(\mathbf{C})\right).$$

Translation invariance allows us to replace $\mathcal{F}(D(z_i, r_i)) \simeq \mathcal{F}(D(0, r_i))$ which we denote by $\mathcal{F}(r_i)$, so we can write this map as

$$\mu^{r_1,\ldots,r_k}_{z_1,\ldots,z_k} : \mathcal{F}(r_1) \times \cdots \times \mathcal{F}(r_k) \to \mathrm{Hol}\left(\mathrm{Disks}(r_1, \ldots, r_k), H^*\mathcal{F}(\mathbf{C})\right).$$

Note that although the source space of this map does not depend on the centers of the disks, the map itself does, hence the messy notation.

For $r' < r$ the maps $\mu^{r_1,\ldots,r_k}_{z_1,\ldots,z_k}$ respect the natural inclusions

$$\mathrm{Disks}(r_1, \ldots, r_k) \hookrightarrow \mathrm{Disks}(r'_1, \ldots, r'_k)$$

and so the limit of the multiplication map as $(r_1, \ldots, r_k) \to (0, \ldots, 0)$ makes sense and has the form

$$\begin{aligned}\mu_{z_1,\ldots,z_k} : \left(\lim_{r\to 0} H^*(\mathcal{F}(r))\right)^{\otimes k} &\to \lim_{r\to 0} \mathrm{Hol}\left(\mathrm{Disks}_k(r), H^*(\mathcal{F}(r))\right) \\ &\cong \mathrm{Hol}\left(\mathrm{Conf}_k(\mathbf{C}), H^*\mathcal{F}(\mathbf{C})\right)\end{aligned}$$

where $\mathrm{Conf}_k(\mathbf{C})$ is the ordered configuration space of k-distinct points in $\mathbf{C}$.

The last piece of data we need corresponds to the "conformal decomposition" of a vertex algebra. For us, this will come from an S^1-action on $\mathcal{F}$. The reader is encouraged to look at [9] for a precise definition, but we assume that we have a *nice* action of S^1 on $\mathcal{F}$ and it is compatible with the translation invariance discussed above.

We can now read off the data of the vertex algebra from $\mathcal{F}$:

– Let $\mathcal{F}^{(l)}(r) \subset \mathcal{F}(r)$ be the lth eigenspace for the S^1-action. The underlying vector space for the vertex algebra is

$$V := \bigoplus_l H^*(\mathcal{F}^{(l)}(r)).$$

Note that the right-hand side depends, a priori, on r. The assumptions of Theorem 7 below ensure that up to isomorphism it is actually independent of r, so we leave the radius out of our notation.

– The translation operator. The action of ∂_z on $\mathcal{F}^{(l)}(r)$ has the form

$$\partial_z : \mathcal{F}^{(l)}(r) \to \mathcal{F}^{(l-1)}(r).$$

We let $T : V \to V$ be the operator which is ∂_z restricted to the l-th eigenspace.

– The fields. Consider the map

$$\mu_{z,0} : \left(\lim_{r\to 0} \mathrm{H}^*(\mathcal{F}(r))\right)^{\otimes 2} \to \mathrm{Hol}\left(\mathrm{Conf}_2(\mathbf{C}), H^*(\mathcal{F}(\mathbf{C}))\right)$$

defined above. Certainly, we have a map $V \to \lim_{r\to 0} \mathrm{H}^*(\mathcal{F}(r))$, where the limit denotes the inverse limit of vector spaces, so it makes sense to restrict $\mu_{z,0}$ to a map

$$V \otimes V \to \mathrm{Hol}\left(\mathrm{Conf}_2(\mathbf{C}), \mathrm{H}^*(\mathcal{F}(\mathbf{C}))\right) \simeq \mathrm{Hol}\left(\mathbf{C}^\times, \mathrm{H}^*(\mathcal{F}(\mathbf{C}))\right).$$

Post composing this with the projection maps $H^*(\mathcal{F}(\infty)) \to V_l$ combine to define the map

$$\bar{\mu}_{z,0} : V \otimes V \to \prod_l \mathrm{Hol}(\mathbf{C}^\times, V_l)$$

We can perform Laurent expansions to view this as

$$\bar{\mu}_{z,0} : V \otimes V \to \bar{V}[[z^{\pm 1}]].$$

We define $Y(-, z) : V \to \mathrm{End}(V)[[z^{\pm 1}]]$ by

$$Y(v, z)v' := \bar{\mu}_{z,0}(v, v').$$

One can show that this actually lies in $V((z))$ for all v, v'.

The above can be made much more precise and made into the following theorem.

Theorem 7 (Theorem 5.2.2.1 [9]) *Let $\mathcal{F}$ be a S^1-equivariant holomorphically translation-invariant prefactorization algebra on* $\mathbf{C}$. *Suppose*

- *The action of S^1 on $\mathcal{F}(r)$ extends smoothly to an action of the algebra of distributions on S^1.*
- *For $r < r'$ the map*

$$\mathcal{F}^{(l)}(r) \to \mathcal{F}^{(l)}(r')$$

  is a quasi-isomorphism.
- *The cohomology $H^*(\mathcal{F}^{(l)}(r))$ vanishes for $l \gg 0$.*
- *For each l and $r > 0$ we require that $H^*(\mathcal{F}^{(l)}(r))$ is isomorphic to a countable sequential colimit of finite-dimensional vector spaces.*

Then $\mathbb{V}\mathrm{ert}(\mathcal{F}) := \oplus_l H^*(\mathcal{F}^{(l)}(r))$ (which is independent of r by assumption) has the structure of a vertex algebra.

Let $\mathrm{PreFact}_{\mathbf{C}}$ denote the category of prefactorization algebras on $\mathbf{C}$. Let $\mathrm{PreFact}_{\mathbf{C}}^{\mathrm{hol}} \subset \mathrm{PreFact}_{\mathbf{C}}$ be the full subcategory spanned by prefactorization algebras satisfying the conditions of the above theorem. This result can be upgraded to provide a functor

$$\mathbb{V}\mathrm{ert} : \mathrm{PreFact}_{\mathbf{C}}^{\mathrm{hol}} \to \mathrm{Vert}$$

where Vert is the category of vertex algebras.

4.3 Verifying the axioms

In this section, we verify the Virasoro factorization algebra $\mathcal{V}\mathrm{ir}$ indeed satisfies the conditions of Theorem 7 necessary to determine a vertex algebra stated in the last section.

Explicitly, we show the following:

(1) There is a S^1-action on $\mathcal{V}\mathrm{ir}$ covering the action of $\mathbf{C}^\times$ by rotations. Moreover, for all $r > 0$ (including $r = \infty$) the S^1-action on $\mathcal{V}\mathrm{ir}(r)$ extends to an action of $\mathcal{D}(S^1)$ the space of smooth distributions on the circle.

(2) Then for all l and all $r < r'$ the natural map

$$\mathcal{V}\mathrm{ir}^{(l)}(r) \to \mathcal{V}\mathrm{ir}^{(l)}(r')$$

is an equivalence.

(3) $H^*(\mathcal{V}\mathrm{ir}^{(l)}(r)) = 0$ for $l \gg 0$.

(4) The space $H^*(\mathcal{V}\mathrm{ir}^{(l)}(r))$ is a colimit of finite-dimensional vector spaces for all l, r.

The first condition is clear: the S^1-action comes from its natural action on $\Omega_c^{0,*}(\mathbf{C})$. We extend this to distributions $\varphi \in \mathcal{D}(S^1)$ by the rule

$$(\varphi \cdot \alpha)(z) = \int_{t \in S^1} \varphi(t)\alpha(tz)$$

where $\alpha \in \Omega_c^{0,*}(\mathbf{C})$. This extends naturally to vector fields.

Let's consider (2). For simplicity, we work with the (untwisted) factorization algebra $\mathcal{V}\mathrm{ir}_0 = C_*(\mathcal{L}_{\mathbf{C}})$; the twisted case is similar. Consider the filtration of $\mathcal{V}\mathrm{ir}_0$ by symmetric tensor degree. Namely

$$F^m \mathcal{V}\mathrm{ir}_0(r) = \mathrm{Sym}^{\leq m}(\mathcal{L}(D(0,r)[1]) = \bigoplus_{j \leq m} \left(\mathcal{L}(D(0,r))[1]^{\otimes j}\right)_{\Sigma_j}.$$

The associated graded of this filtration is

$$\mathrm{Gr}^m \mathcal{V}\mathrm{ir}_0(r) = \left(\mathcal{L}(D(0,r))[1]^{\otimes m}\right)_{\Sigma_m}$$

and there is a spectral sequence

$$H^*(\mathrm{Gr}^* \mathcal{V}\mathrm{ir}_0(r)) \Rightarrow H^*(\mathcal{V}\mathrm{ir}_0(r)).$$

The filtration respects the S^1-action, so for each l we get a spectral sequence for the eigenspaces

$$H^*(\mathrm{Gr}^* \mathcal{V}\mathrm{ir}_0^{(l)}(r)) \Rightarrow H^*(\mathcal{V}\mathrm{ir}_0^{(l)}(r)).$$

Thus, to verify that $\mathcal{V}\mathrm{ir}_0^{(l)}(r) \to \mathcal{V}\mathrm{ir}_0^{(l)}(s)$ is an equivalence for $r < s$, it enough to show that it is at the level of associated gradeds. That is, we need to show that the restriction of the map

$$\Omega_c^{0,*}\left(D(0,r)^m, TD(0,r)^{\boxtimes m}\right) \to \Omega_c^{0,*}\left(D(0,s)^m, TD(0,s)^{\boxtimes m}\right)$$

to the l-eigenspaces is an equivalence. Again, we recall Serre's result that for any open $U \subset \mathbf{C}$

$$H^*_{\bar{\partial}}\left(\Omega_c^{0,*}(U, TU)\right) \cong \left(\Omega^1_{\mathrm{hol}}(U, TU)\right)^\vee$$

concentrated in degree 0. When $U = D(0,r)$ we have a coordinatization

$$\Omega^1_{\mathrm{hol}}(D(0,r)) = \mathbf{C}[z]\mathrm{d}z.$$

Now, z^k has S^1-weight k. Thus $(z^k)^\vee$ has weight $-k$. The weight of $(\mathrm{d}z)^\vee$ is -1 and the weight of $\partial_z^\vee$ is $+1$. This shows that the weight spaces are independent of the radius chosen, so we have verified (2). Moreover, the weight spaces are clearly finite dimensional and vanish for $m \geq 0$, so we also get (3) and (4).

Finally, Theorem 7 implies the following.

Proposition 8 *The $\mathbf{C}[c]$-module $V = \bigoplus_l H^*(\mathcal{V}\mathrm{ir}^{(l)}(r))$ has the structure of the vertex algebra (in $\mathbf{C}[c]$-modules) induced from the factorization structure on $\mathcal{V}\mathrm{ir}$. In particular, for each $c \in \mathbf{C}$ the vector space $V_c = \bigoplus_l \mathrm{H}^*(\mathcal{V}\mathrm{ir}_c^{(l)}(r))$ has the structure of a vertex algebra.*

4.4 An isomorphism of vertex algebras

The map $\Phi : U(\mathrm{Vir}) \to H^*(\mathcal{V}\mathrm{ir}(A(r, r')))$ from Proposition 4 applied to the interval $I = (r, r')$ gives V the structure of a $U(\mathrm{Vir})$-module. More precisely, let $\epsilon < r < R$ then we have a factorization map

$$\mathcal{V}\mathrm{ir}(D(0, \epsilon)) \otimes \mathcal{V}\mathrm{ir}(A(r, R)) \to \mathcal{V}\mathrm{ir}(D(0, R)).$$

We have the following diagram

$$\begin{tikzcd}
H^*\mathcal{V}\mathrm{ir}(D(0,\epsilon)) \otimes H^*\mathcal{V}\mathrm{ir}(A(r,R)) \arrow[rr] & & H^*\mathcal{V}\mathrm{ir}(D(0,R)) \\
V \otimes H^*\mathcal{V}\mathrm{ir}(A(r,R)) \arrow[u] & & V \arrow[u] \\
V \otimes U(\mathrm{Vir}) \arrow[u, "1\otimes\Phi_c"] \arrow[urr, dotted] & &
\end{tikzcd}$$

The top left arrow comes from the inclusion $V \hookrightarrow \mathrm{H}^*(\mathcal{V}\mathrm{ir}(D(0, \epsilon)))$. The dotted map exists since the image of the factorization product on V, where we only see finite sums of S^1-eigenvectors, still only contains finite sums of S^1-eigenvectors.

Our main result is:

Theorem 9 *There is a $\mathbf{C}[c]$-linear isomorphism of $U(\mathrm{Vir})$-modules $\Psi : \mathbf{Vir} \to V$ which sends $|0\rangle \in \mathbf{Vir}_c$ to $1 \in V$. It extends to an isomorphism of vertex algebras*

$$\Psi : \mathbf{Vir} \xrightarrow{\cong} \mathbb{V}\mathrm{ert}(\mathcal{V}\mathrm{ir})$$

over the ring $\mathbf{C}[c]$. In particular, when we specialize to a $c \in \mathbf{C}$ we obtain an isomorphism of vertex algebras

$$\Psi_c : \mathbf{Vir}_c \xrightarrow{\cong} \mathbb{V}\mathrm{ert}(\mathcal{V}\mathrm{ir}_c).$$

Proof Recall that the vacuum vector is the image of 1 under the map

$$U(\mathrm{Vir}) \xrightarrow{\mathrm{id}\otimes 1} U(\mathrm{Vir}) \otimes \mathbf{C} \longrightarrow U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir})_+} \mathbf{C}_c = \mathbf{Vir}_c.$$

We define the map of $U(\mathrm{Vir})$-modules

$$U(\mathrm{Vir}) \otimes \mathbf{C} \to V$$

by sending $1 \otimes 1$ to 1 and extending by $U(\mathrm{Vir})$-linearity. We need to check that this descends to $\mathbf{Vir}_c$. That is, we verify that $1 \in V$ is killed by L_n for $n \geq -1$. Recall $L_n(A) = f(z\bar{z})z^{n+2}\mathrm{d}\bar{z}\partial$ is a representative for L_n on $\mathcal{L}^{\mathbf{C}}(A(r, R))$ where $f(z\bar{z})$ is a bump function as above. It suffices to show that $L_n(A)$ is exact when viewed as an element in $\mathcal{L}^{\mathbf{C}}(D(0, R))$. Define $h(z, \bar{z}) := \int_{z\bar{z}}^{\infty} f(s)\, \mathrm{d}s$ and note that the chain rule implies

$$\bar{\partial}(h(z, \bar{z})\, z^{n+1}) \;=\; f(z\bar{z})z^{n+2}\mathrm{d}\bar{z}$$

Thus, $L_n(A)$ is exact via the element $h(z, \bar{z})z^{n+1}\partial$. This shows that we get a well-defined map $\mathbf{Vir}_c \to V$ that sends $|0\rangle \mapsto 1$.

We need to see that this map is an isomorphism of $U(\mathrm{Vir})$-modules. We take advantage of some filtrations. Consider the natural filtration of the tensor algebra of Vir, namely

$$F^i(\mathrm{Tens}(\mathrm{Vir})) := \otimes_{j\geq i}\mathrm{Vir}.$$

This descends to a filtration on $U(\mathrm{Vir})$. Similarly, define the filtration of $\mathcal{V}\mathrm{ir}$ by

$$F^i\mathcal{V}\mathrm{ir}(U) = \mathrm{Sym}^{\leq i}(\Omega_c^{0,*}(U, TU)[1] \oplus \mathbf{C}\cdot C)$$

with induced differential. It is clear that this is a subcomplex and hence descends to cohomology. Moreover, the map of modules defined above respects both of these filtrations.

With respect to the above filtration, we have the identification

$$\mathrm{Gr}\ \mathbf{Vir} \cong \mathrm{Sym}^*(z^{-1}\mathbf{C}[z^{-1}]\partial_z \oplus \mathbf{C}\cdot C) = \mathrm{Sym}^*(z^{-1}\mathbf{C}[z^{-1}]\partial_z)[C].$$

Moreover, we have the identifications of associated gradeds

$$\begin{aligned}\mathrm{Gr}\ U(\mathrm{Vir}) \;&=\; \mathrm{Sym}(\mathrm{Vir}) = \mathrm{Sym}(\mathbf{C}[z, z^{-1}]\partial_z)[C] \;\text{ and }\; \mathrm{Gr}\ \mathcal{V}\mathrm{ir}(U)\\ &= \hat{\mathrm{Sym}}(\Omega_c^{0,*}(U, TU)[1] \oplus \mathbf{C}\cdot C).\end{aligned}$$

Consider the map $U(\mathrm{Vir}) \to V$ induced by the action of $U(\mathrm{Vir})$ on the unit $1 \in V$. We have the diagram of associated gradeds

$$\begin{tikzcd}
\mathrm{Gr}\ U(\mathrm{Vir}) \arrow[r] \arrow[d, equal] & \mathrm{Gr}\ V \arrow[r, hook] & \mathrm{H}^*\left(\mathrm{Sym}^*(\Omega_c^{0,*}(D(0,r), TD(0,r))[1] \oplus \mathbf{C}\cdot C)\right) \\
\mathrm{Sym}(\mathbf{C}[z, z^{-1}]\partial_z)[C] \arrow[ur] & &
\end{tikzcd}$$

The embedding on the right is the direct sum of S^1-eigenspaces and is identified with $z^{-1}\mathbf{C}[z^{-1}][C]$. Thus, the map Gr $U(\mathrm{Vir}) \to$ Gr V is the map of commutative algebras

$$\mathrm{Sym}(\mathbf{C}[z, z^{-1}]\partial_z)[C] \to \mathrm{Sym}(z^{-1}\mathbf{C}[z^{-1}]\partial_z)[C]$$

and is induced by natural map $\mathbf{C}[z, z^{-1}] \to z^{-1}\mathbf{C}[z^{-1}]$.

Concluding, we see that the map Gr $\mathbf{Vir} \to$ Gr V is an isomorphism, and since there are no extension problems over $\mathbf{C}[c]$, we have the desired isomorphism of $U(\mathrm{Vir})$-modules.

Finally, we need to show that the OPE's agree so that the module isomorphism extends to an isomorphism of vertex algebras. Namely, we will show

$$m_{z,0}(L_{-2} \cdot 1, v) = \sum_{n \in \mathbf{Z}} (L_n \cdot v) z^{-n-2}.$$

Now, the residue pairing allows us to represent $L_n(A(r, R))$ by the linear map

$$\Omega^1_{\mathrm{hol}}(A(r, R)) \to \mathbf{C} \ , \quad h(z)\mathrm{d}z \mapsto \left(\oint_{S^1} z^{n+1} h(z) \mathrm{d}z \right).$$

Fix a point $z_0 \in A(r, R)$. By Cauchy's theorem, we have for some ϵ such that $\epsilon < |z_0| - r$ and $\epsilon < s - |z_0|$:

$$2\pi i h(z_0) = \oint_{|\zeta|=R-\epsilon} \frac{h(\zeta)}{\zeta - z_0} d\zeta - \oint_{|\zeta|=r+\epsilon} \frac{h(\zeta)}{\zeta - z_0} d\zeta.$$

For the first integral, we have $|z_0| < |\zeta|$ and we can expand

$$\frac{1}{\zeta - z_0} = \frac{1}{\zeta} \cdot \frac{1}{1 - \frac{z_0}{\zeta}} = \frac{1}{\zeta} \sum_{j=0}^{\infty} \left(\frac{z_0}{\zeta} \right)^j = \sum_{j=0}^{\infty} z_0^j \zeta^{-j-1}.$$

Thus

$$\oint_{|\zeta|=R-\epsilon} \frac{h(\zeta)}{\zeta - z_0} d\zeta = \sum_{j=0}^{\infty} \left(\oint_{|\zeta|=R-\epsilon} h(\zeta) \zeta^{-j-1} d\zeta \right) z_0^j.$$

Similarly, the second integral can be written as

$$\oint_{|\zeta|=r+\epsilon} \frac{h(\zeta)}{\zeta - z_0} d\zeta = -\sum_{j=0}^{\infty} \left(\oint_{|\zeta|=r+\epsilon} h(\zeta) \zeta^j \right) z_0^{-j-1}.$$

Since h is holomorphic on $A(r, R)$, we can combine these integrals by choosing a common contour and reindexing to write

$$\sum_{j=0}^{\infty}\left(\oint_{|\zeta|=R-\epsilon} h(\zeta)\zeta^{-j-1}d\zeta\right) z_0^j + \sum_{j=0}^{\infty}\left(\oint_{|\zeta|=r+\epsilon} h(\zeta)\zeta^j\right) z_0^{-j-1}$$
$$= \sum_{n\in\mathbf{Z}}\left(\oint \zeta^{n+1}h(\zeta)d\zeta\right) z_0^{-n-2}.$$

This completes the proof.

5 Universal factorization algebras and the Virasoro

It is a natural to extend the Virasoro factorization algebra to general one-dimensional complex manifolds. Moreover, from the point of view of conformal field theory, [20] for instance, it is essential to consider this a global version of the Virasoro algebra defined on general Riemann surfaces. Vertex algebras are of course local in nature, from above they correspond to factorization on $\mathbf{C}$. In this section, we transition to studying a version of the Virasoro factorization algebra defined on a general one-dimensional complex manifold.

One approach would be to construct a factorization algebra on each manifold independently. It is convenient for us, however, to consider the *site* of complex manifolds. Define the category Hol_1 whose objects are one-dimensional complex manifolds and whose maps are holomorphic embeddings. This is a symmetric monoidal category with respect to disjoint union $\sqcup$. Just as in the case of a fixed manifold, Weiss covers define a Grothendieck topology on Hol_1.

Definition 5.1 A *universal* holomorphic prefactorization algebra (valued in the category $\mathrm{dgNuc}^{\otimes}$) is a symmetric monoidal functor

$$\mathrm{Hol}_1^{\sqcup} \to \mathrm{dgNuc}^{\otimes}.$$

A *universal* holomorphic factorization algebra is a universal holomorphic prefactorization algebra satisfying descent for Weiss covers.

Remark 7 The term *universal* has appeared in the literature of vertex algebras and their close relatives, chiral algebras, and we'd like to point out how our terminology is different. In Section 3.4.14 of [2] the term *universal chiral algebra* is used to refer to chiral algebras that are valued in the category of modules for the Harish–Chandra pair $(\mathrm{Aut}(\hat{D}), W_1)$ of formal automorphisms and formal derivations of the holomorphic disk. In Section 6.3 of [13] such a structure in the category of vertex algebras is referred to as a *quasi-conformal* vertex algebra. We stress that this is different than the notion of universal considered in Definition 5.1. One can also realize the analog of a quasi-conformal structure in the setting of holomorphic factorization algebras by factorization algebras valued in the category of $(\mathrm{Aut}(\hat{D}), W_1)$ modules.

More recently, the concept of universal chiral algebras has appeared in [5,6] which studies the relationship between factorization algebras in the sense of Beilinson–Drinfeld that are universal with respect to étale maps (rather than holomorphic embeddings) and factorization algebras in the appropriate category of Harish–Chandra modules.

We can produce such universal holomorphic factorization algebras from sheaves of Lie algebras on the site Hol_1. Indeed, given a sheaf of Lie algebras $\mathcal{G}$ we can apply the Chevalley–Eilenberg chains functor applied to compactly supported sections $C_*(\mathcal{G}_c)$ to get a universal factorization algebra. Moreover, this functor satisfies descent so that it defines a universal holomorphic factorization algebra. We will denote this universal factorization algebra by $U^{\mathrm{fact}}\mathcal{G}$.

Example 4 Let us consider a fundamental example of a universal holomorphic factorization algebra. Fix an ordinary Lie algebra $\mathfrak{g}$ and define the sheaf of Lie algebras on Hol_1 by sending the complex one-manifold Σ to the dg Lie algebra $\mathfrak{g}^{\Sigma} := \Omega^{0,*}(\Sigma; \mathfrak{g})$. The differential is given by $\bar{\partial} \otimes 1_{\mathfrak{g}}$ and the Lie bracket extends that of $\mathfrak{g}$. In doing so, one obtains the universal factorization algebra $U^{\mathrm{fact}}\mathfrak{g}^{(-)}$ that sends $\Sigma \mapsto C_*(\mathfrak{g}^{\Sigma})$. If $\mathfrak{g}$ has a invariant pairing $\langle -, - \rangle_{\mathfrak{g}}$ one can use the cocycle on $\mathfrak{g}^{\Sigma}$ defined by

$$(\alpha, \beta) \mapsto \int_{\Sigma} \langle \alpha \wedge \partial\beta \rangle_{\mathfrak{g}}$$

to define a central extension $\hat{\mathfrak{g}}^{\Sigma}$. One obtains a universal factorization algebra via

$$U^{\mathrm{fact}}\hat{\mathfrak{g}} : \Sigma \mapsto C_*(\hat{\mathfrak{g}}^{\Sigma}).$$

This is the universal factorization algebra representing the Kac-Moody vertex algebra, see Chapter 5 of [9].

We will produce the universal Virasoro factorization algebra in the same manner. Indeed, for each Σ in Hol_1 we have the dg Lie algebra

$$\mathcal{L}^{\Sigma} = \Omega^{0,*}(\Sigma, T_{\Sigma})$$

with differential given by $\bar{\partial}$ and bracket given by extending the usual Lie bracket of holomorphic vector fields. The assignment $\mathcal{L} : \Sigma \mapsto \mathcal{L}^{\Sigma}$ defines a symmetric monoidal functor from the category Hol_1 to the category of dg Lie algebras with symmetric monoidal structure given by direct sum (of underlying graded vector spaces). As the functor of Chevalley–Eilenberg chains $C_*(-)$ is symmetric monoidal, we get a symmetric monoidal functor given by the *universal* envelope of $\mathcal{L}$

$$U^{\mathrm{fact}}\mathcal{L} : \mathrm{Hol}_1 \to \mathrm{dgNuc}^{\otimes} \quad , \quad \Sigma \mapsto C_*(\mathcal{L}_c^{\Sigma}).$$

Applied to $\Sigma = \mathbf{C}$, of course we are in the situation of the previous portion of the paper.

The interesting part from the point of view of conformal field theory and representation theory is the envelope of a *central extension* of the sheaf of dg Lie algebras $\mathcal{L}$. There is a potential problem defining this central extension based on our formula given in Sect. 2. Indeed, the cocycle on $\mathcal{L}^{\mathbf{C}}$

$$\omega(\alpha \otimes \partial_z, \beta \otimes \partial_z) = \frac{1}{2\pi}\frac{1}{12}\int_U \left(\partial_z^3 \alpha_0 \beta_1 + \partial_z^3 \alpha_1 \beta_0\right) \mathrm{d}^2 z$$

clearly depends on the choice of a coordinate (its failure to be coordinate independent is precisely measured by the Schwarzian). Thus, there is no obvious way of constructing a universal twisted envelope on all holomorphic one-manifolds simultaneously.

5.1 First fix: uniformization

A Riemann surface is a complex manifold of dimension one. Therefore, it is given by a covering $\{U_i\}$ such that all transition functions are holomorphic diffeomorphisms. The cocycle ω is not invariant under arbitrary diffeomorphisms: if $w = f(z)$, it is not necessarily true that $f^*(\omega_z) = \omega_w$.

One way of formulating the uniformization theorem for Riemann surfaces is that one can always find a subordinate cover to $\{U_i\}$ such that the transition functions have the form

$$w = f(z) = \frac{az + b}{cz + d}$$

with $ad - bc \neq 0$. I.e., we can reduce to the projective linear structure group. Let $\mathrm{Hol}_1^{\mathrm{proj}} \subset \mathrm{Hol}_1$ denote the full subcategory of covers where the transition functions are projective. The above says that there is a section $\mathrm{unif} : \mathrm{Hol}_1 \to \mathrm{Hol}_1^{\mathrm{proj}}$ of the inclusion

$$\mathrm{Hol}_1 \hookrightarrow \mathrm{Hol}_1^{\mathrm{proj}}.$$

Lemma 3 *The cocycle ω is invariant under projective changes of coordinate. That is, for f a projective diffeomorphism one has $f^*\omega = \omega$.*

Thus, we can form a factorization algebra $\mathcal{F}_\omega^{\mathrm{Vir}}$ on $\mathrm{Hol}_1^{\mathrm{proj}}$. Using the uniformization construction, this pulls back to a factorization algebra on Riemann surfaces via

$$\mathrm{Hol}_1 \xrightarrow{\mathrm{unif}} \mathrm{Hol}_1^{\mathrm{proj}} \xrightarrow{\mathcal{F}_\omega^{\mathrm{Vir}}} \mathrm{dgNuc}.$$

The problem with this construction is that the induced extension cocycle is not so obvious to write down. There is a more explicit way of doing this.

5.2 Second fix: projective connections

We recall Atiayh's [1] formulation of connections on holomorphic vector bundles. Let E be a holomorphic vector bundle on a complex manifold X. Denote by $\mathrm{Diff}^{\leq 1}(E) \subset \mathrm{Diff}(E)$ the subspace of order one differential operators on E. There is a short exact sequence of vector bundles

$$0 \to \mathrm{End}(E) \to \mathrm{Diff}^{\leq 1}(E) \to T_X^{1,0} \otimes \mathrm{End}(E) \to 0$$

where the last map is the symbol map of an order one differential operator. Form the pull-back along the inclusion of $T_X^{1,0} \hookrightarrow T_X^{1,0} \otimes \mathrm{End}(E)$ via $x \mapsto x \otimes \mathrm{id}$. The resulting bundle is the *Atiyah*-bundle

$$0 \to \mathrm{End}(E) \to \mathrm{At}(E) \to T_X^{1,0} \to 0.$$

Atiyah showed that splittings of this sequence are precisely holomorphic connections.

Consider the inclusion $\mathcal{O}_X \hookrightarrow \mathrm{End}(E)$ by viewing $s \mapsto f \cdot s$ for $f \in \mathcal{O}_X$. One gets the induced sequence of bundles

$$0 \to \mathrm{End}(E)/\mathcal{O}_X \to \mathrm{At}(E)/\mathcal{O}_X \to T_X^{1,0} \to 0.$$

By definition, *projective connections* are splittings of the above sequence.

- Non-trivial holomorphic connections on T_Σ exist only in genus 1, this is a consequence of Riemann–Roch.
- Projective connections on T_Σ exist for all Riemann surfaces and form a torsor over quadratic holomorphic differentials $\Omega^1_{\mathrm{hol}}(\Sigma)^{\otimes 2}$.

Let Hol_1^∇ denote the category of pairs (Σ, ∇) where ∇ is a projective connection for the holomorphic tangent bundle $T_\Sigma^{1,0}$. There is a forgetful functor

$$\pi : \mathrm{Hol}_1^\nabla \to \mathrm{Hol}_1$$

that we should think of as a $(\Omega^1_{hol})^{\otimes 2}$-torsor.

Fix a projective connection ∇ on Σ. Locally, on Σ consider the bilinear on $\mathcal{L}_c(U_z)$

$$\omega_{\nabla,z}(X, Y) = \omega_z(X, Y) + \nabla_z \cdot [X, Y].$$

Proposition 10 – *ω_∇ defines a cocycle on $\mathcal{L}_c(U_z)$ and is invariant under holomorphic changes of coordinate.*
- *If ∇' is another projective connection, we have $\omega_\nabla \sim \omega_{\nabla'}$.*

Proof **Coordinate invariance** In writing down $\omega_{\nabla,z} = \omega_z$ we have used a coordinate. We check coordinate invariance, so that it defines a section over Σ. It suffices to understand the case $U_z = \mathbf{C}_z$. Suppose $f : \mathbf{C}_w \to \mathbf{C}_z$ is a change of coordinates. Let $u : \mathcal{L}^{\mathbf{C}} \otimes \mathcal{L}^{\mathbf{C}} \to \Omega^{1,1}_{\mathbf{C}}$ denote the bilinear map

$$(\alpha \partial_z, \beta \partial_z) \mapsto \left(\partial_z^3 \alpha_0 \beta_1 - \alpha_0 \partial_z^3 \beta_1) + (\partial_z^3 \alpha_1 \beta_0 - \alpha_1 \partial_z^3 \beta_0)\right) \mathrm{d}\bar{z} \mathrm{d}z.$$

We compute the difference

$$f^*u(\alpha\partial_z, \beta\partial_z) - u\left(f^*(\alpha\partial_z), f^*(\beta\partial_z)\right) = 2\left[(\alpha_0\partial_w\beta_1 - \partial_w\alpha_0\beta_1) + (\alpha_1\partial_w\beta_0 - \partial_w\alpha_1\beta_0)\right] S(f)\left(\frac{\overline{\partial f}}{\partial w}\right)\mathrm{d}w\mathrm{d}\bar{w}$$

where $S(f)$ is the holomorphic function called the Schwarzian. Explicitly, it is given in terms of first, second, and third holomorphic derivatives of f:

$$S(f)(z) = \frac{\partial}{\partial z}\left(\frac{\partial^2 f/\partial z^2}{\partial f/\partial z}\right) - \frac{1}{2}\left(\frac{\partial^2 f/\partial z^2}{\partial f/\partial z}\right)^2.$$

So the failure of the cocycle u to be independent of a choice of coordinate is measured by the Schwarzian.

Let $P : \mathcal{L}^{\mathbf{C}} \otimes \mathcal{L}^{\mathbf{C}} \to \Omega^{1,1}(\mathbf{C})$ be the bilinear

$$(\alpha\partial_z, \beta\partial_z) \mapsto \rho_z \cdot ((\alpha_0\partial_z\beta_1 - \partial_z\alpha_0\beta_1) + (\alpha_1\partial_z\beta_0 - \partial_z\alpha_1\beta_0))\, \mathrm{d}z\mathrm{d}\bar{z}.$$

We compute the difference

$$f^*P(\alpha\partial_z, \beta\partial_z) - P\left(f^*(\alpha\partial_z), f^*(\beta\partial_z)\right) = ((\alpha_0\partial_z\beta_1 - \partial_z\alpha_0\beta_1) + (\alpha_1\partial_z\beta_0 - \partial_z\alpha_1\beta_0))\, S(f)\left(\frac{\overline{\partial f}}{\partial w}\right)\mathrm{d}w\mathrm{d}\bar{w}.$$

This shows that the bilinear $u + 2P$ is independent of choice of coordinates. Finally, note that

$$\omega_{\mathbf{C}} = \int_{\mathbf{C}} \circ\, (u + 2P)$$

is the desired cocycle defining the extension of $\mathcal{L}^{\mathbf{C}}$, so we are done.

Cocycle condition We need to show that ω_U is a cocycle for the Lie algebra $\mathcal{L}^{\Sigma}(U)$ for all U. We suppose $U \simeq \mathbf{C}$ and we check $\omega_{\mathbf{C}}$ is a cocycle. For simplicity write elements $\alpha\partial \in \mathcal{L}^{\mathbf{C}}(\mathbf{C})$ as α. In terms of the bilinears u, P above we have

$$\begin{aligned}
&\omega_{\mathbf{C}}([\alpha, \beta], \gamma) + \omega_{\mathbf{C}}([\beta, \gamma], \alpha) + \omega_{\mathbf{C}}([\gamma, \alpha], \beta) \\
&= \int_{\mathbf{C}} \big[(u([\alpha, \beta], \gamma) + u([\beta, \gamma], \alpha) + u([\gamma, \alpha], \beta)) \\
&\quad +2\, (P([\alpha, \beta], \gamma) + P([\beta, \gamma], \alpha) + P([\gamma, \alpha], \beta))\big]
\end{aligned}$$

It follows from Jacobi that P-terms vanish. So, it suffices to show that

$$\int_{\mathbf{C}} (u([\alpha, \beta], \gamma) + u([\beta, \gamma], \alpha) + u([\gamma, \alpha], \beta)) = 0.$$

This is a straightforward calculation.

Now, we show independence of $\omega_{\nabla,z}$ on a projective connection. Again, this is a local calculation. Suppose ∇, ∇' are two projective connections, and let ω, ω' and P, P' denote induced the bilinears as above, respectfully. We need to show that $\omega - \omega'$ is a coboundary when viewed as a cocycle in $C^*_{\mathrm{red}}(\mathcal{L}^{\mathbf{C}})$. As mentioned above, the difference of two ordinary projective connections is simply a quadratic differential. It follows that we may view the difference $\nabla - \nabla'$ as an element in $\Omega^{0,*}(\Sigma, K_\Sigma^{\otimes 2})$. Then, we see that for $X \in \mathcal{L}^\Sigma$

$$(\nabla - \nabla') \cdot X = \left\langle \nabla - \nabla', X \right\rangle$$

where $\langle -, - \rangle$ denotes the natural pairing

$$\Omega^{0,*}(\Sigma, K_\Sigma^{\otimes 2}) \otimes \Omega^{0,*}(\Sigma, T_\Sigma) \to \Omega^{0,*}(\Sigma, K_\Sigma) \simeq \Omega^{1,*}(\Sigma).$$

Denote by $\Phi = \int \circ \left\langle \nabla - \nabla', - \right\rangle_{(1,1)} : \mathcal{L}^{\mathbf{C}} \to \mathbf{C}$. Note that Φ is linear of degree -1, so that it is a 0-cocycle for $\mathcal{L}^{\mathbf{C}}$. We have

$$(\omega - \omega')(\alpha, \beta) = \Phi([\alpha, \beta]).$$

This is what we wanted to show.

We take away two main observations: (1) there is a local cocycle $\omega \in H^1_{\mathrm{loc}}(\mathcal{L}^\Sigma)$ for each Σ and hence an associated factorization algebra $\tilde{\mathcal{V}}\mathrm{ir}$ on Hol_1^∇ and (2) that we can descend along π to get a factorization algebra $\mathcal{V}\mathrm{ir} : \mathrm{Hol}_1 \to \mathrm{dgNuc}$ as desired. When restricted to the over category of open sets $U \subset \mathbf{C}$, we produce the factorization algebra from the first part of the paper, hence the repetition of notation.

The prefactorization algebra $\mathcal{V}\mathrm{ir}$ can be described explicitly as follows. To a pair (Σ, ∇) of a Riemann surface together with a projective connection, we define

$$\tilde{\mathcal{V}}\mathrm{ir}(\Sigma, \nabla) = U^{\mathrm{fact}} \hat{\mathcal{L}}^\Sigma = U^{\mathrm{fact}}_{\omega_\Delta} \mathcal{L}^\Sigma$$

where $\hat{\mathcal{L}}^\Sigma$ is the dg Lie algebra that is the extension of $\mathcal{L}^\Sigma$ determined by the cocycle ω_∇.

Thus, the factorization algebra $\mathcal{V}\mathrm{ir}$ on Hol_1 has the following interpretation. Given a Riemann surface Σ choose *any* projective connection ∇. Form the twisted envelope as above. By the proposition, this extension is independent of the projective connection chosen.

5.3 Fixed Riemann surface

For each Riemann surface Σ, we can restrict our factorization algebra $\mathcal{V}\mathrm{ir}$ to the overcategory $\mathrm{Hol}_{1/\Sigma}$ to get a factorization algebra on Σ which we denote $\mathcal{V}\mathrm{ir}^\Sigma$. The construction depends on the 2-cocycle ω, but on a fixed Riemann surface the choice is unique up to a scaling.

Proposition 11 *Let Σ be any Riemann surface. Then, we have*

$$H^1\left(C^*_{\mathrm{loc}}(\mathcal{L}^\Sigma)\right) \;\cong\; \mathbf{C}.$$

Proof Consider $\Sigma = \mathbf{C}$. Then

$$C^*_{\mathrm{loc}}(\mathcal{L}_{\mathbf{C}}) \;=\; \Omega^*_{\mathbf{C}} \otimes_{\mathcal{D}_{\mathbf{C}}} C^*_{\mathrm{red}}\left(\mathrm{Jet}_{\mathcal{L}_{\mathbf{C}}}\right) \;\cong\; \Omega^*\left(\mathbf{C}; C^*_{\mathrm{red}}(\mathrm{Jet}_{\mathcal{L}_{\mathbf{C}}})\right)[2]$$

Now, $\mathrm{Jet}_{\mathcal{L}_{\mathbf{C}}}$ corresponds to the dg Lie algebra $\mathbf{C}[[z, \bar{z}, \mathrm{d}\bar{z}]]\partial_z$ with differential given by $\bar{\partial}$. This is quasi-isomorphic to the Lie algebra of formal holomorphic vector fields $W_1 := \mathbf{C}[[z]]\partial$ (with zero differential). So, we see $C^*_{\mathrm{loc}}(\mathcal{L}_{\mathbf{C}}) \simeq C^*(W_1)[2]$. A calculation of Gelfand-Fuchs [14] implies that

$$H^*_{\mathrm{red}}(W_1) = \mathbf{C}[-3]$$

concentrated in degree 3. The generator is of the form $\partial_z^\vee \cdot (z\partial_z)^\vee \cdot (z^2\partial_z)^\vee$.

We'd like to bootstrap this to the global case. Consider the filtration of $\Omega^*\left(\Sigma, C^*_{\mathrm{red}}(\mathrm{Jet}_{\mathcal{L}^\Sigma})\right)$ by form degree. This spectral sequence has E_2-page

$$E_2 = \Omega^*\left(\Sigma, \underline{H}^*\left(C^*_{\mathrm{red}}(\mathrm{Jet}_{\mathcal{L}^\Sigma}\right)\right).$$

Here, $\underline{H}$ means the cohomology $\mathcal{D}$-module. We have computed the cohomology of the fibers of $\underline{H}^*\left(C^*_{\mathrm{red}}(\mathrm{Jet}_{\mathcal{L}^\Sigma})\right)$, and they are concentrated in a single degree. Choosing a formal coordinate at a point in Σ trivializes the fiber of this point to $\mathbf{C}\left\langle\partial_z^\vee \cdot (z\partial_z)^\vee \cdot (z^2\partial_z)^\vee\right\rangle$. This trivialization is independent of coordinate choice and compatible with the flat connection. Thus

$$\underline{H}^*(C^*_{\mathrm{red}}(\mathrm{Jet}_{\mathcal{L}^\Sigma}) \simeq C^\infty_\Sigma[-3]$$

with its usual flat connection. This completes the proof.

5.4 Symmetries by vector fields

The primary appearance of the Virasoro vertex algebra in physics is as a symmetry of two-dimensional conformal field theories. That is, the Virasoro vertex algebra acts on conformal field theories with a specified central charge. Later on we will see an example of how the Virasoro factorization algebra appears as a symmetry of certain holomorphic quantum field theories using the BV formalism as developed in [9,10]. For now, we would like to discuss the meaning of such a Virasoro symmetry on general holomorphic factorization algebras.

A vertex algebra is conformal of central charge c if there is an element $v_c \in V$ such that the Fourier coefficients L_n^V of the vertex operator $Y(v_c, z) = \sum_n L_n^V z^{-n-2}$ span a Lie algebra that is isomorphic to Vir_c. Moreover, one requires that $L_{-1}^V = T$ the translation operator, and $L_0^V|_{V_n} = n \cdot \mathrm{Id}_{V_n}$. This can be wrapped up by saying we have a map of vertex algebras $\mathbf{Vir}_c \to V$ sending $L_{-2} \cdot 1$ to v_c.

Motivated by this, we introduce the following terminology for holomorphic factorization algebras. We say a Virasoro symmetry of central charge c of a holomorphic factorization algebra $\mathcal{F}$ is a map of holomorphic factorization algebras

$$\Phi : \mathcal{V}\mathrm{ir}_c \to \mathcal{F}. \tag{4}$$

Remark 8 A holomorphic factorization algebra is a symmetric monoidal functor

$$\mathcal{F} : \mathrm{Hol}_1 \to \mathrm{dgNuc}$$

A map of holomorphic factorization algebras is a natural transformation between symmetric monoidal functors of the above form. In particular, in the definition above we require the existence of maps

$$\Phi(\Sigma) : \mathcal{V}\mathrm{ir}_c(\Sigma) \to \mathcal{F}(\Sigma)$$

for each one-dimensional complex manifold Σ. Moreover, these maps must be natural with respect to holomorphic embeddings.

In the next section, we will show an example using BV quantization to implement a map of factorization algebras $\mathcal{V}\mathrm{ir}_c \to \mathcal{F}$ where $\mathcal{F}$ is the quantum observables of the $\beta\gamma$ system. In the remainder of this section we'd like to extract one consequence of having a Virasoro symmetry of charge c. We see that in the case of factorization algebras on $\mathbf{C}$ we recover the usual notion of a conformal vertex algebra.

Indeed, suppose that $\mathcal{F}$ is a holomorphic factorization algebra on $\mathbf{C}$ satisfying the conditions of Theorem 7. Then a Virasoro symmetry of central charge c from (4) induces the structure of a conformal vertex algebra on $\mathbb{V}\mathrm{ert}(\mathcal{F})$ of charge c. As the construction $\mathbb{V}\mathrm{ert}(\mathcal{F})$ is functorial, we obtain a map of vertex algebra

$$\mathbb{V}\mathrm{ert}(\Phi) : \mathrm{Vir}_c \to \mathbb{V}\mathrm{ert}(\mathcal{F}).$$

Explicitly, the conformal vector is given by $\mathbb{V}\mathrm{ert}(\Phi)(L_{-2}1_{\mathrm{Vir}}) \in \mathbb{V}\mathrm{ert}(\mathcal{F})$.

A fundamental object in conformal field theory is the so-called bundle of conformal blocks on the moduli space of curves. Given a vertex algebra describing a holomorphic conformal field theory the action of the Virasoro Lie algebra is necessary for the construction of bundle equipped with a projectively flat connection through a process called "Virasoro uniformization", see Chapter 17 of [13], for instance. This is a version of Gelfand-Kazhdan descent (sometimes referred to as Harish–Chandra localization) along a certain bundle of coordinates over the pointed moduli of curves $\mathcal{M}_{g,1}$ which we briefly summarize. The moduli space $\mathcal{M}_{g,1}$ consists of pairs (Σ, x) where Σ is a curve and $x \in \Sigma$. There exists a canonically defined Vir_0-torsor $\hat{\mathcal{M}}_{g,1}$ over $\mathcal{M}_{g,1}$ consisting of triples (Σ, x, φ) where φ is a formal coordinate near x. One then considers modules for the *pair* $(\mathrm{Aut}(\hat{D}), \mathrm{Vir}_0)$ where $\mathrm{Aut}(\hat{D})$ is the group automorphisms of the formal disk. These objects are simultaneously modules for Vir_0 and the group $\mathrm{Aut}(\hat{D})$ that are compatible with the natural inclusion of Lie algebras $\mathrm{Lie}(\mathrm{Aut}(\hat{D})) \hookrightarrow \mathrm{Vir}_0$. In practice, and in all of the examples we care about, Vir_0-modules can be exponentiated to modules for the pair.

If one starts with a module V for the pair $(\mathrm{Aut}(\hat{D}), \mathrm{Vir}_0)$ Virasoro uniformization can be viewed as a two-step process. First, one forms the associated bundle over $\mathcal{M}_{g,1}$ using the action of formal automorphisms. The residual action of Vir_0 defines the data of a flat connection. In the case that one has an action of Vir_c, for some nonzero charge c, one gets a *projectively flat* connection. The resulting object is no longer a D-module on the moduli of curves, but rather a module for a sheaf of twisted differential operators. In the case that V is a conformal vertex algebra, the resulting bundle is the bundle of conformal blocks equipped with its projectively flat connection. For instance, in the case of the Virasoro vertex algebra of central c, one finds a sheaf of twisted differential operators on $\mathcal{M}_{g,1}$ (see [3], for instance).

One can attempt a similar construction at the level of factorization algebras. Indeed, let $\mathcal{F}$ be a holomorphic factorization algebra on $\mathbf{C}$ that is equipped with a map of factorization algebras

$$\Phi : \mathcal{V}\mathrm{ir}_c \to \mathcal{F}$$

as in (4). In the case that $\mathcal{F}$ is holomorphic, we see that $\mathcal{F}(D)$ is a module for the Lie algebra of annular observables. Indeed, we have a factorization map

$$\mu : \mathcal{F}(A) \otimes \mathcal{F}(D) \to \mathcal{F}(D_{big})$$

where D_{big} is a disk centered at zero containing the annulus A. We have already seen that the structure maps coming from nested annuli give $\mathcal{F}(A)$ the structure of a Lie algebra. Suppose that for any inclusions of disks centered at zero $D(0, r) \hookrightarrow D(0, R)$ the induced map

$$\mathcal{F}(D(0, r)) \xrightarrow{\simeq} \mathcal{F}(D(0, R))$$

is a quasi-isomorphism. Then, the structure map μ together with the map $H^*\Phi(A)$: $\mathrm{Vir}_c \simeq H^*(\mathcal{V}\mathrm{ir}_c(A)) \to \mathcal{F}(A)$ give $H^*(\mathcal{F})$ the structure of a module over the Lie algebra Vir_c. Thus, we can descend the space $\mathcal{F}(D)$ to get a sheaf equipped with a projective flat connection on $\mathcal{M}_{g,1}$. One expects that the fiber of this bundle over a fixed curve Σ coincides with the global sections, or factorization homology $\int_\Sigma \mathcal{F}$ defined in the next section.

6 Factorization homology and correlation functions

6.1 Global sections

In this section, we compute the cohomology of the global sections of the factorization algebra $\mathcal{V}\mathrm{ir}^\Sigma$. This is known as the factorization homology of $\mathcal{V}\mathrm{ir}^\Sigma$ and is denoted by

$$\int_\Sigma \mathcal{V}\mathrm{ir}^\Sigma = H^*(\mathcal{V}\mathrm{ir}^\Sigma(\Sigma)).$$

In the language of chiral algebras, the cohomology of global sections is often referred to as the "chiral homology" in the literature [2,12] and is dual to the space of "conformal blocks". We will discuss conformal blocks for the Virasoro in more detail shortly.

Remark 9 As noted above $\mathrm{B}\mathcal{L}^{\Sigma}$ describes the formal completion at Σ inside of $\mathcal{M}_g$, the moduli of Riemann surfaces of genus g. We have already remarked that $\int_{\Sigma} \mathcal{V}\mathrm{ir}^{\Sigma}$ is the ∞-jet at Σ of a certain sheaf on the moduli of curves $\mathcal{M}_g$, namely the sheaf of twisted differential operators. An independent definition of this sheaf on the moduli of curves from the point of view of factorization algebras is non-trivial, and we defer making any precise relationships at the moment.

Now, we compute the factorization homology. We will need the following fact about the Dolbeault resolution of holomorphic vector fields which is immediate from studying the cohomology of Riemann surfaces for each g.

Proposition 12 *The dg Lie algebra* $(\Omega^{0,*}(\Sigma, T_{\Sigma}), \bar{\partial})$ *is formal for any Riemann surface* Σ.

That is, the dg Lie algebras $\Omega^{0,*}(\Sigma, T_{\Sigma})$ and $H^*_{\bar{\partial}}(\Omega^{0,*}(\Sigma, T_{\Sigma}))$ are quasi-isomorphic. It follows that $H^*(\mathrm{Vir}^{\Sigma}(\Sigma))$ is equal to the cohomology of the complex

$$\left(\mathrm{Sym}(\mathrm{H}^*(\Sigma, T\Sigma) \oplus \mathbf{C} \cdot C), d_{\mathrm{CE}}\right)$$

since $\bar{\partial}$ kills the central term C.

The full differential on $\mathcal{V}\mathrm{ir}^{\Sigma}$ is $\bar{\partial} + \mathrm{d}_{\mathrm{Lie}} + \omega$ where $\mathrm{d}_{\mathrm{Lie}}$ is the Chevalley–Eilenberg differential for the Lie algebra $\Omega^{0,*}(\Sigma, T\Sigma)$ and ω is the cocycle of Sect. 5.3.

The case $g = 0$

We have $H^*(\Sigma_0, T_{\Sigma_0}) \cong \mathfrak{sl}_2(\mathbf{C})$ generated by the vector fields $\partial_z, z\partial_z$, and $z^2\partial_z$. For degree reasons, the central extension does not contribute to the Lie differential. Thus

$$\int_{\Sigma_0} \mathcal{V}\mathrm{ir}^{\Sigma_0} \;\cong\; \mathrm{H}^{\mathrm{Lie}}_*(\mathfrak{sl}_2(\mathbf{C}))[c] \;\cong\; \mathbf{C}[y, C]$$

with $\deg(y) = 3$ and $\deg(C) = 0$.

The case $g = 1$

In this case, we know that the dg Lie algebra $\mathrm{H}^*(\Sigma_1, T_{\Sigma_1}) = \mathbf{C} \oplus \mathbf{C}[-1]$ with zero Lie bracket and zero differential. Moreover, H^0 is generated by the constant vector field ∂_z. The bilinear form defining the central extension vanishes on constant vector fields so doesn't contribute to the Lie differential. Thus

$$\int_{\Sigma_1} \mathcal{V}\mathrm{ir}^{\Sigma_1} \;\cong\; \mathrm{Sym}(\mathbf{C}[1] \oplus \mathbf{C} \oplus \mathbf{C} \cdot C) \;\cong\; \mathbf{C}[x, y, C]$$

with $\deg(x) = -1$ and $\deg(y) = \deg(C) = 0$.

The case $g > 1$

The dg Lie algebra is $H^*(\Sigma_g, T_{\Sigma_g}) = \mathbf{C}^{3g-3}[-1]$. For degree reasons, this algebra is abelian and does not interact with the central extension. Thus

$$\int_{\Sigma_g} \mathcal{V}\mathrm{ir}^{\Sigma_g} \cong \mathrm{Sym}(\mathbf{C}^{3g-3} \oplus \mathbf{C} \cdot C) \cong \mathbf{C}[y_1, \dots, y_{3g-3}, C]$$

with $\deg(y_1) = \cdots = \deg(y_{3g-3}) = \deg(c) = 0$.

6.2 Correlation functions

In this section, we compute the *correlation functions* associated to the Virasoro factorization algebra. These calculations are reminiscent for those of the conformal blocks of a conformal vertex algebra and exhibits the utility of our approach to factorization in CFT.

Fix a Riemann surface Σ and consider a collection of disjoint opens $U_1, \dots, U_n \subset \Sigma$. The n-point correlation function for associated to these open sets is the factorization structure map

$$\Phi_{U_1,\dots,U_N} : \mathcal{V}\mathrm{ir}^{\Sigma}(U_1) \otimes \cdots \otimes \mathcal{V}\mathrm{ir}^{\Sigma}(U_n) \to \mathcal{V}\mathrm{ir}^{\Sigma}(\Sigma).$$

Consider the case of $\Sigma = \mathbf{C}$ and suppose that each of the opens U_i is a biholomorphic to a disk of a certain fixed radius r. Suppose, moreover, that $\mathcal{F}$ is a holomorphically translation-invariant factorization algebra on $\mathbf{C}$. Then it is an algebra over the cooperad $\Omega^{0,*}(\mathrm{Disks})$. In particular, for each n we can think of the n-point correlator as a holomorphic function on the space

$$\mathrm{Disks}_n(r) \simeq \mathrm{Conf}_n(\mathbf{C}).$$

We now describe an explicit way of calculating these n-point correlation functions that bears some resemblance to the standard method of computing correlation functions in conformal field theory.

First, we fix a partial inverse $\bar{\partial}^{-1}$ for the Dolbeault operator $\bar{\partial}$ for the holomorphic tangent bundle T_Σ. This operator vanishes on harmonic functions and 1-forms and is inverse to $\bar{\partial}$ on the complement to the space of harmonic functions and 1-forms. We can construct it as follows. Let G be a Green's function for the $\bar{\partial}$ operator. It satisfies the equation

$$\bar{\partial} G = \omega_{\mathrm{diag}}$$

where ω_{diag} is the $(1, 1)$-form on $\Sigma \times \Sigma$ that is the volume element along the diagonal and zero elsewhere. Given G we define the operator $\bar{\partial}^{-1}$ via the formula

$$(\bar{\partial}^{-1}\varphi)(z) = \int_w G(z, w)\varphi(w).$$

Let $a_1, \dots, a_n \in \Omega^{0,*}(\Sigma, T\Sigma)$ be $\bar\partial$ closed. We will write down a general formula for the cohomology class of the factorization product $a_1 \cdots a_n$. Moreover, suppose that a_1 is in the orthogonal complement to harmonic $(0, *)$ forms. Then consider the expression

$$
\begin{aligned}
(\bar\partial + \mathrm{d}_{\mathrm{Lie}} + \omega)\left((\bar\partial^{-1}a_1)a_2\cdots a_n\right) = {} & (\bar\partial\bar\partial^{-1}a_1)a_2\cdots a_n \\
& + \sum_{j=2}^{n}(-1)^{j+1}\left[\bar\partial^{-1}a_1, a_j\right]a_2\cdots\hat{a}_j\cdots a_n \\
& + \sum_{j=2}^{n}(-1)^{j+1}\omega(\bar\partial^{-1}a_1, a_j)a_2\cdots\hat{a}_j\cdots a_n.
\end{aligned}
$$

The first line follows from the fact that the only non-trivial Lie bracket involving the elements $a_1, \dots, a_n$ is between a_1 and a_j for $j \neq 1$. The second line follows from the fact that the cocycle ω is cohomologically degree one.

Since the term on the left hand side is exact in the cochain complex $\mathcal{V}\mathrm{ir}^{\Sigma}(\Sigma)$, we have at the level of cohomology

$$
\begin{aligned}
\lfloor a_1\cdots a_n\rfloor = {} & \sum_{j=2}^{n}(-1)^{j}\left\lfloor[\bar\partial^{-1}a_1, a_j]a_2\cdots\hat{a}_j\cdots a_n\right\rfloor \\
& + \sum_{j=2}^{n}(-1)^{j}\omega(\bar\partial^{-1}a_1, a_j)\left\lfloor a_2\cdots\hat{a}_j\cdots a_n\right\rfloor. \qquad (5)
\end{aligned}
$$

In particular, we see that $\lfloor a\rfloor = 0$ for any a.

6.2.1 Genus zero

We can use this formula to recover well-known relations involving the genus zero correlation functions. Fix a collection of points $(x_1, \dots, x_n) \in \mathrm{Conf}_n(\mathbf{C}P^1)$ and suppose $\epsilon > 0$ is such that the collection of disks $\{D(x_i, \epsilon)\}$ are pairwise disjoint. Fix radial bump functions $f_{x_i}(z, \bar z) = f(r^2)$ for the disks $D(x_i, \epsilon)$ and consider the $(0, 1)$-forms $f_{x_i}(z, \bar z)\mathrm{d}\bar z \in \Omega^{0,1}(D(x_i, \epsilon))$ which define the holomorphic vector field valued forms

$$
a_{x_i}(z, \bar z) := f_{x_i}(z, \bar z)\mathrm{d}\bar z\partial_z \in \Omega^{0,1}\left(D(x_i, \epsilon), T\mathbf{C}P^1|_{D(x_i,\epsilon)}\right) \subset \mathcal{V}\mathrm{ir}(D(x_i, \epsilon))
$$

on $D(x_i, \epsilon)$. One should think of a_{x_i} as a mollified version of a point-like observable supported at x_i.

We will compute the resulting n-point correlation functions

$$
\left\lfloor a_{x_1}\cdots a_{x_n}\right\rfloor \in H^0\int_{\mathbf{C}P^1}\mathcal{V}\mathrm{ir} \cong \mathbf{C}\cdot C.
$$

Here, we note that each a_{x_i} is a (linear) degree zero element in the factorization algebra $\mathcal{V}ir$ so the resulting element in factorization homology is also degree zero. We have already computed that H^0 of the factorization homology on $\mathbf{C}P^1$ is one-dimensional spanned by the central element C.

Using the explicit form of the operator $\bar{\partial}^{-1}$ on $\mathbf{C}P^1$, we find

$$\bar{\partial}^{-1}(a_{x_i}(z,\bar{z})) = \frac{1}{z-x_1}\partial_z.$$

For $a_i = a_{x_i}$, the recursive equation for the n-point function Eq. (5) becomes

$$\begin{aligned}\lfloor a_{x_1}\cdots a_{x_n}\rfloor &= \sum_{j=2}^{n}(-1)^j \left\lfloor \left[\frac{1}{z-x_1}\partial_z, a_{x_j}(z,\bar{z})\right] a_{x_2}\cdots\hat{a}_j\cdots a_{x_n}\right\rfloor \\ &\quad + c\sum_{j=2}^{n}(-1)^j\omega\left(\frac{1}{z-x_1}\partial_z, f_j(z,\bar{z})\mathrm{d}\bar{z}\partial_z\right)\lfloor a_{x_2}\cdots\hat{a}_{x_j}\cdots a_{x_n}\rfloor.\end{aligned}$$

Let us use this formula to compute the n-point function for small n. We have already remarked that $\lfloor a_{x_1}\rfloor = 0$. Now, suppose $x_1 \neq x_2$, then the recursive formula implies

$$\lfloor a_{x_1}a_{x_2}\rfloor = c\omega\left(\bar{\partial}^{-1}a_{x_1}, a_{x_2}\right).$$

By definition of the cocycle ω, the right-hand side is equal to

$$c\cdot\frac{1}{12}\int_z \frac{1}{z-x_1}\partial_z^3(f_{x_2}(z,\bar{z}))\mathrm{d}z\mathrm{d}\bar{z}.$$

Iterative application of integration by parts together with the fact that $\int\varphi(z)f_{x_2}(z,\bar{z})\mathrm{d}z\mathrm{d}\bar{z} = \varphi(x_2)$ yields

$$\lfloor a_{x_1}a_{x_2}\rfloor = \frac{c}{2}\frac{1}{(x_1-x_2)^4}.$$

We can compute $\lfloor a_{x_1}a_{x_2}a_{x_3}\rfloor$ in a similar way. Since $\lfloor a_{x_i}\rfloor = 0$ the recursive formula implies

$$\lfloor a_{x_1}a_{x_2}a_{x_3}\rfloor = \left\lfloor\left[\frac{1}{z-x_1}\partial_z, a_{x_2}(z,\bar{z})\right]\cdot a_{x_3}\right\rfloor - \left\lfloor\left[\frac{1}{z-x_1}\partial_z, a_{x_3}(z,\bar{z})\right]\cdot a_{x_2}\right\rfloor. \tag{6}$$

Consider the first term above. We compute the Lie bracket

$$\left[\frac{1}{z-x_1}\partial_z, a_{x_2}\right] = \frac{1}{z-x_1}\partial_z(f_{x_2}(z,\bar{z}))\mathrm{d}\bar{z}\partial_z + \frac{1}{(z-x_1)^2}f_{x_2}(z,\bar{z})\mathrm{d}\bar{z}\partial_z.$$

Applying $\bar{\partial}^{-1}$ to this expression yields the vector field

$$\left(-\frac{1}{(z-x_2)^2(x_2-x_1)}+\frac{2}{(z-x_2)(x_2-x_1)^2}\right)\partial_z.$$

This calculation, combined with the fact that $\lfloor a\cdot b\rfloor = c\omega(\bar{\partial}^{-1}a, b)$ yields

$$\begin{aligned}\left\lfloor\left[\frac{1}{z-x_1}\partial_z, a_{x_2}(z,\bar{z})\right]\cdot a_{x_3}\right\rfloor &= -c\omega\left(\frac{1}{(z-x_2)^2(x_2-x_1)}\partial_z, a_{x_3}(z,\bar{z})\right)\\ &\quad +2c\omega\left(\frac{1}{(z-x_2)(x_2-x_1)^2}\partial_z, a_{x_3}(z,\bar{z})\right)\\ &= -\frac{c}{12}\int_z\frac{1}{(z-x_2)^2(x_2-x_1)}\partial_z^3(f_{x_3}(z,\bar{z}))\mathrm{d}z\mathrm{d}\bar{z}\\ &\quad +\frac{c}{6}\int_z\frac{1}{(z-x_2)(x_2-x_1)^2}\partial_z^3(f_{x_3}(z,\bar{z}))\mathrm{d}z\mathrm{d}\bar{z}\\ &= \frac{c}{(x_3-x_2)^4(x_2-x_1)}\left(-\frac{2}{x_3-x_2}+\frac{1}{x_2-x_1}\right)\end{aligned}$$

The second term in (6) is obtained by sending $x_2 \leftrightarrow x_3$ in the above formula. In total, the sum is thus

$$\frac{c}{(x_3-x_2)^4(x_2-x-1)(x_3-x_1)}\left(-2\frac{x_3-x_2}{x_3-x_2}+\frac{x_3-x_1}{x_2-x_1}+2\frac{x_2-x_1}{x_3-x_2}+\frac{x_2-x_1}{x_3-x_1}\right).$$

This simplifies to the following expression for the 3-point correlator

$$\lfloor a_{x_1}a_{x_2}a_{x_3}\rfloor = \frac{c}{(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2}.$$

For general n, the recursive formula implies that can write the n-point function as

$$\begin{aligned}\lfloor a_{x_1}\cdots a_{x_n}\rfloor &= \sum_{j=2}^{n}\left(\frac{1}{x_j-x_1}\partial_{x_j}+\frac{1}{(x_j-x_1)^2}\right)\lfloor a_{x_2}\cdots a_{x_n}\rfloor\\ &\quad +\frac{c}{2}\sum_{j=2}^{n}(-1)^j\frac{1}{(x_j-x_1)^4}\lfloor a_{x_2}\cdots\hat{a}_{x_j}\cdots a_{x_n}\rfloor.\end{aligned}$$

This shows, in particular, that as a function on the space $\mathrm{Conf}_n(\mathbf{C}P^1)$ the correlation function is not only holomorphic, it is rational. One can find this expression for the correlation functions in the vertex algebra literature, see for instance Section 2 of [22].

7 Application: Virasoro symmetry for holomorphic factorization algebras

7.1 Example: the $\beta\gamma$ system

In this section we'd like to explain an example of a family of holomorphic field theories parametrized by an integer n whose factorization algebra of observables has the structure of a holomorphic factorization algebra that we denote Obs_n^q with a Virasoro symmetry. We produce a map of factorization algebras on $\mathbf{C}$ from the Virasoro factorization algebra (at a certain central charge) to Obs_n^q. As a corollary we show that we recover the usual Virasoro vector of the $\beta\gamma$ vertex algebra.

First we need to define the factorization algebra Obs_n^q. First, we define a precosheaf of dg Lie algebras. To a one-dimensional Riemannian manifold U, we define the dg Lie algebra

$$\mathcal{H}_n(U) := \Omega_c^{1,*}(U)^{\oplus n} \oplus \Omega_c^{0,*}(U)^{\oplus n} \oplus \mathbf{C}[-1]$$

with bracket given by

$$[\varphi, \psi] := \sum_{i=1}^{n} \int_U \varphi_i \wedge \psi_i .$$

Here we write each element in components as $\varphi = (\varphi_1, \dots, \varphi_n) \in \Omega_c^{*,*}(U)^{\oplus n}$ so that $\varphi_i \in \Omega_c^{*,*}(U)$. The factorization algebra is obtained in a similar way to the envelope of a local Lie algebra. To an open set U, we define $\mathrm{Obs}_n^q(U) := C_*^{\mathrm{Lie}}(\mathcal{H}_n(U))$. Explicitly

$$\mathrm{Obs}_n^q(U) := \left(\mathrm{Sym}\left(\Omega_c^{1,*}(U)^{\oplus n}[1] \oplus \Omega_c^{0,*}(U)^{\oplus n}[1]\right), \bar{\partial} + \Delta\right) \tag{7}$$

where Δ is the Chevalley–Eilenberg differential coming from the Lie bracket. We restrict ourselves to factorization algebras on the Riemann surface $\mathbf{C}$.

It is shown in [9] that Obs_1^q is a holomorphic factorization algebra whose associated vertex algebra is isomorphic, to the one-dimensional $\beta\gamma$ vertex algebra. Similarly, one has the following

Theorem 13 [[9] Theorem 5.3.3.2] *The vertex algebra $\mathbb{V}\mathrm{ert}(\mathrm{Obs}_n^q)$ is isomorphic to the n-dimensional $\beta\gamma$ vertex algebra V_n. The vertex algebra V_n has state space spanned by vectors $\{b_l^i, c_m^j\}$ where $l < 0$, $m \leq 0$ and $1 \leq i, j \leq n$ with vertex operators*

$$Y(b_{-1}^i, z) = \sum_{l<0} b_m^i z^{-1-n} + \sum_{l \geq 0} \frac{\partial}{\partial c_{-l}^i} z^{-1-n}$$

$$Y(c_0^j, z) = \sum_{m \leq 0} c_m^j z^{-m} - \sum_{m>0} \frac{\partial}{\partial b_{-m}^j} z^{-m} .$$

Proposition 14 *There is a map of factorization algebras on* $\mathbf{C}$

$$\Phi : \mathcal{V}\mathrm{ir}^{\mathbf{C}}_{c=n} \to \mathrm{Obs}^q_n$$

commuting with the S^1 *action. This map quantizes the map of factorization algebras*

$$\Phi^{cl} : \mathcal{V}\mathrm{ir}^{\mathbf{C}}_{c=0} \to \mathrm{Obs}^{cl}_n.$$

In particular, the map of factorization algebras produces a map of vertex algebras

$$\mathbb{V}\mathrm{ert}(\Phi) : \mathbb{V}\mathrm{ert}(\mathcal{V}\mathrm{ir}^{\mathbf{C}}_{c=n}) \to \mathbb{V}\mathrm{ert}(\mathrm{Obs}^q_n).$$

Concluding the proof of the proposition, we will see explicitly that the map of vertex algebras produced by the above proposition recovers the usual conformal vector of the $\beta\gamma$ vertex algebra.

7.1.1 A remark on quantization

In the statement of the above proposition, we claimed that the map Φ is a quantization of Φ^{cl}. On the one hand, a consequence of this is that in the limit as $\hbar \to 0$ the map Φ equals Φ^{cl}. There is, however, a more refined version of what quantization means here.

In the book [9,10] a *quantization* of a factorization algebra refers to a $\hbar$-deformation of a Poisson bracket on the factorization algebra, analogous to the usual picture from deformation quantization. In this context, however, the bracket is of cohomological degree 1 and gives the classical factorization algebra a P_0-*structure*. In the usual story of deformation quantization the deformed object has the structure of an associative algebra. For us, the quantum factorization algebras have the structure of a Beilinson-Drinfeld (BD) algebra. For a precise definition of P_0 and BD structured factorization algebras see Sections 2.3 and 2.4 of [10].

Thus, implicit in the statement of quantization in Proposition 14 is that the "classical" factorization algebras $\mathcal{V}\mathrm{ir}^{\mathbf{C}}_{c=0}$ and Obs^{cl}_n have a P_0-structure. The P_0 structure on Obs^{cl}_n is clear, it comes from the (-1)-shifted symplectic pairing defining the classical $\beta\gamma$ system. The P_0 structure on $\mathcal{V}\mathrm{ir}^{\mathbf{C}}_{c=0}$ is more subtle, and we will describe it now.

Note that for every open set U we can write the underlying graded vector space of the factorization algebra evaluated on U as

$$\mathcal{V}\mathrm{ir}^{\mathbf{C}}_{c=0}(U)^{\#} = \mathrm{Sym}\left(\bar{\Omega}^{1,*}(U, T^*U)\right)^{\vee}$$

where the bar denotes distributional sections. To check this, we use the relationship $\left(\bar{\Omega}^{1,*}(U, T^*U)\right)^{\vee} \cong \Omega^{0,*}_c(U, TU)[1]$ where $(-)^{\vee}$ denotes the continuous dual as above. Thus, we can interpret $\mathcal{V}\mathrm{ir}^{\mathbf{C}}_{c=0}(U)$ as the space of functions on the formal moduli space $B\left(\bar{\Omega}^{1,*}(U, T^*U)[-1]\right)$. We will realize the P_0 structure on $\mathcal{V}\mathrm{ir}^{\mathbf{C}}_{c=0}$ as coming from a shifted Poisson tensor on $B(\bar{\Omega}^{1,*}(U, T^*U)[-1])$. For conventions of degrees and notations of P_0 structures on formal moduli problems, we refer the reader to [4].

The P_0 structure has two pieces, a constant term and a linear term. First, consider the Lie bracket

$$[-,-] : \Omega_c^{0,*}(U, TU) \otimes \Omega_c^{0,*}(U, TU) \to \Omega_c^{0,*}(U, TU).$$

Its continuous dual is a map of the form

$$[-,-]^\vee : \bar{\Omega}^{1,*}(U, T^*U) \to \bar{\Omega}^{1,*}(U, T^*U) \otimes \bar{\Omega}^{1,*}(U, T^*U)$$

and defines the linear piece of the P_0 structure. One checks immediately that it is of the correct degree. Next, we remark that the continuous dual cocycle

$$\omega : \Omega_c^{0,*}(U, TU) \otimes \Omega_c^{0,*}(U, TU) \to \mathbf{C}[-1]$$

introduced in Sect. 2.2.2 can be regarded as an element in $\mathrm{Sym}^2(\bar{\Omega}^{1,*}(U, T^*U))$ and determines the constant term of the P_0 structure. Again, it is immediate to check that it is of the correct degree. The data defining the P_0 structure was defined in terms of differential operators of the underlying graded vector bundles. It follows, Proposition 2.22 of [4], that the P_0 structure is compatible with the factorization product of $\mathcal{V}\mathrm{ir}_{c=0}^{\mathbf{C}}$ and hence defines a P_0 structured factorization algebra. A small extension of this provides a P_0 structured factorization algebra $\mathcal{V}\mathrm{ir}_{c=0}^{\Sigma}$ for any surface Σ.

7.2 Proof of Proposition

The proof is based on an explicit calculation in terms of Feynman diagrams in a version of renormalization developed in [8] and [10].

To describe the map in Proposition 14, it is necessary to describe the factorization algebra Obs_n^q in terms of an effective family of factorization algebras and functionals as in [10]. The general formalism starts with a classical field theory defined by a symplectic form of cohomological degree -1 and produces from an effective quantization, as in [8], a factorization algebra of quantum observables.

The fields of the theory are

$$\mathcal{E}_n := \Omega^{0,*}(\mathbf{C})^{\oplus n} \oplus \Omega^{1,*}(\mathbf{C})^{\oplus n}.$$

We write the fields as (γ, β) (hence the name), and the components as $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\beta = (\beta_1, \ldots, \beta_n)$. The symplectic pairing is

$$\langle \gamma, \beta \rangle = \sum_{i=1}^{n} \int_{\mathbf{C}} \gamma_i \beta_i$$

which is easily seen to have cohomological degree -1. With this pairing, we can express $\mathcal{E}_n$ as $\Omega^{0,*}(\mathbf{C}) \otimes V \oplus \Omega^{1,*}(\mathbf{C}) \otimes V^*$ where V is a complex n-dimensional vector space. The pairing comes from the dual pairing on V.

The classical observables supported on Σ are simply the space of algebraic functions on the space of fields. Keeping track of the right notion of duals, for any open $U \subset \mathbf{C}$ we define

$$\bar{\mathrm{Obs}}_n^{cl}(\mathbf{C}) = \mathrm{Sym}\left(\bar{\Omega}_c^{1,*}(\mathbf{C})^{\oplus n} \oplus \bar{\Omega}_c^{0,*}(\mathbf{C})^{\oplus n}\right).$$

One checks immediately that this construction defines a factorization algebra on $\mathbf{C}$.

The classical action of holomorphic vector fields is a very natural one. Given any element $\alpha \in \Omega^{*,*}(U)$ and any section of the holomorphic tangent bundle $X \in \Gamma(U, T^{1,0}U)$, we define

$$X \cdot \alpha = L_X \alpha$$

where $L_X\alpha$ denotes the Lie derivative of α by X. This definition naturally extends to elements $X \in \mathcal{L}^{\mathbf{C}}(U) = \Omega^{0,*}(U, T^{1,0}U)$. This action of $\mathcal{L}^{\mathbf{C}}$ on forms leads to an action of the factorization algebra $\bar{\mathrm{Obs}}_n^{cl}$ as follows. For $X \in \mathcal{L}^{\mathbf{C}}$ define the holomorphically translation-invariant local functional $I_X^{\mathcal{L}} \in \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_n)$ by

$$I_X^{\mathcal{L}}(\gamma, \beta) = \int \beta \wedge (X \cdot \gamma).$$

Note that this local functional is of cohomological degree -1 and so we have described a map

$$I^{\mathcal{L}} : \mathcal{L}^{\mathbf{C}} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_n)[-1].$$

The space of local functionals shifted up by one $\mathcal{O}_{\mathrm{loc}}(\mathcal{E}_n)[-1]$ is itself a dg Lie algebra with Lie bracket given by the Poisson bracket $\{-, -\}$ induced from the pairing $\langle -, - \rangle$. It is immediate to check that $I^{\mathcal{L}}$ is compatible with this bracket and hence defines a map of local Lie algebras on $\mathbf{C}$. This implies that $I^{\mathcal{L}}$ determines a Maurer–Cartan element of $C^*_{\mathrm{loc}}(\mathcal{L}^{\mathbf{C}}; \mathcal{O}_{\mathrm{loc}}(\mathcal{E}_n))[-1])$ which we think of as encoding the action of holomorphic vector fields on the classical field theory.

A translation-invariant local functional determines a classical observable supported on any open set in $\mathbf{C}$. For each $U \subset \mathbf{C}$, we then extend $I^{\mathcal{L}}$ to a map of commutative dg algebras $\Phi^{cl}(U) : \mathrm{Sym}(\mathcal{L}_c(U)[1]) \to \bar{\mathrm{Obs}}^{cl}(U)$. These combine to give a map of factorization algebras

$$\Phi^{cl} : \mathcal{V}\mathrm{ir}_0 \to \bar{\mathrm{Obs}}^{cl}.$$

The naive BV Laplacian Δ_0 defined by contraction with the integral kernel K_0 of the symplectic form $\langle -, - \rangle$ is ill-defined on $\bar{\mathrm{Obs}}_n^{cl}$ as it involves pairing distributional sections. This was solved in the above description by working with a smaller class of observables: one defines

$$\mathrm{Obs}_n^{cl}(\mathbf{C}) \subset \bar{\mathrm{Obs}}_n^{cl}(\mathbf{C})$$

to be the subspace of non-distributional sections of the appropriate vector bundles. Then, on Obs_n^{cl} the operator Δ_0 *is* well-defined. Note that when we equip $\mathrm{Obs}_n^{cl}(\mathbf{C})$ with the differential $\bar{\partial} + \Delta_0$, we obtain Obs_n^q as defined in (7). It is immediate that the factorization structures coincide.

There is another solution that is necessary to describe the map in Proposition 14 that involves mollifying the operator Δ_0 to a family of operators Δ_L for each $L > 0$. This approach is outlined in wide generality in Chapter 9 of [10]. In this example, there is an obvious choice on how to mollify Δ_0. Let $\bar{\partial}^*$ be the Hodge dual operator to $\bar{\partial}$ with respect to the Euclidean metric on $\mathbf{C}$. Then the commutator $[\bar{\partial}, \bar{\partial}^*]$ is the Hodge Laplacian. We let $K_{L,n}$ be the integral kernel for the operator $e^{-L[\bar{\partial},\bar{\partial}^*]}$. It is the unique graded symmetric element $K_{L,n} \in \mathcal{E}_n \otimes \mathcal{E}_n$, for $L > 0$, satisfying

$$\left\langle K_{L,n}(z, w), \varphi(w)\right\rangle_w = (e^{-L[\bar{\partial},\bar{\partial}^*]}\varphi)(z)$$

for all $\varphi \in \mathcal{E}_n$. Explicitly, one has $K_{L,n} = K_L \otimes (\mathrm{Id}_V + \mathrm{Id}_{V^*})$ where

$$K_L(z, w) = \frac{1}{4\pi L} e^{-|z-w|^2/4L} (\mathrm{d}\bar{z} \otimes 1 - 1 \otimes \mathrm{d}\bar{w}).$$

The mollified BV Laplacian is the operator Δ_L defined by contraction with K_L. Note that this operator is well-defined on Obs_n^{cl}. The propagator of the theory is $P_\epsilon^L(z, w) \otimes (\mathrm{Id}_V + \mathrm{Id}_{V^*})$ where

$$P_\epsilon^L(z, w) = \int_{t=\epsilon}^L \frac{1}{16\pi t} e^{-|z-w|/4t} \mathrm{d}t.$$

The space of global quantum observables at scale L is the complex

$$\bar{\mathrm{Obs}}_n^q(\mathbf{C})[L] := \left(\mathrm{Sym}\left(\bar{\Omega}_c^{1,*}(\mathbf{C})^{\oplus n} \oplus \bar{\Omega}_c^{0,*}(\mathbf{C})^{\oplus n}\right), \bar{\partial} + \Delta_L\right).$$

To get a factorization algebra structure, we need to provide the space of quantum observables supported on an arbitrary open $U \subset \mathbf{C}$. This is more subtle than in the classical case since the operator Δ_L has support everywhere on the complex line. In fact, to have a reasonable definition we need to consider the BV Laplacian for a more general class of parameterices. This is developed fully in Chapter 8 of [10]. We will not provide details here, as the exact definition of the factorization structure will not be used. The main result we will need is the following.

Proposition 15 *There is a quasi-isomorphism of factorization algebras on* $\mathbf{C}$

$$\mathrm{Obs}_n^q \xrightarrow{\simeq} \bar{\mathrm{Obs}}_n^q$$

where on the right-hand side we use the effective BV quantization provided by the regularized BV operator.

We are given a Maurer–Cartan element $I^{\mathcal{L}}$ that encodes the action of holomorphic vector fields on the classical factorization algebra. Since the field theory underlying the factorization algebra is free, the action lifts to an action of a shifted central extension of holomorphic vector fields. This implies that we have a map of factorization algebras

$$\Phi : U_\alpha \mathcal{L} \to \bar{\mathrm{Obs}}_n^q \tag{8}$$

for some cocycle $\alpha \in C^*_{loc}(\mathcal{L})$ parameterizing the shifted central extension. In the language of effective quantization of BV theories the cocycle α is the $L \to 0$ limit of the obstruction $\alpha[L]$ of the one-loop quantum interaction

$$I^{\mathcal{L}}[L] = \sum_{\Gamma \in \text{Graphs of genus} \leq 1} \frac{1}{|\mathrm{Aut}(\Gamma)|} W_\Gamma(P_0^L \otimes (\mathrm{Id}_V + \mathrm{Id}_{V^*}) \, ; I^{\mathcal{L}})$$

to satisfy the quantum master equation:

$$\bar{\partial} I^{\mathcal{L}}[L] \mathrm{d}_{\mathcal{L}} I^{\mathcal{L}}[L] + \frac{1}{2}\{I^{\mathcal{L}}[L], I^{\mathcal{L}}[L]\}_L + \hbar \Delta_L I^{\mathcal{L}}[L] = \alpha[L].$$

Here, $I^{\mathcal{L}}[L] \in C^*_{\mathrm{Lie}}(\mathcal{L}; \mathcal{O}(\mathcal{E}_n))$ is defined by homotopy RG-flow using the weight expansion in terms of connected graphs of genus less than or equal to one. There is a subtle point, the propogator P_0^L is distributional by nature, so a priori the expression for $I^{\mathcal{L}}[L]$ may not exist. The fact that $I^{\mathcal{L}}[L]$ is well-defined is a hallmark of holomorphic theories having no counter terms when one uses the so-called chiral gauge.

With a calculation similar to that of Corollary 16.0.5 in [7], we have the following description of the effective obstruction cocycle $\alpha[L]$.

Lemma 4 ([7] Corollary 16.0.5)) *The obstruction $\alpha[L]$ is computed by the weight*

$$\lim_{\epsilon \to 0} W_\Gamma(P_\epsilon^L \otimes (\mathrm{Id}_V + \mathrm{Id}_{V^*}) \, , K_\epsilon \otimes (\mathrm{Id}_V + \mathrm{Id}_{V^*}) \, ; I^{\mathcal{L}})$$

where Γ is the one-loop connected wheel with two vertices. We attach the propagator P_ϵ^L to one inner edge and K_L to the other inner edge.

With this lemma in hand, we directly compute $\alpha[L]$. For $X = f(z, \bar{z})\partial_z$ and $g(z, \bar{z})\mathrm{d}\bar{z}\partial_z$ in $\mathcal{L}_c(\mathbf{C})$ we have

$$\alpha[L](f\partial_z, g\mathrm{d}\bar{z}\partial_z) = n \lim_{\epsilon \to 0} \int_{\mathbf{C}_z \times \mathbf{C}_w} f(z, \bar{z}) \left(\partial_z P_\epsilon^L(z, w)\right) g(z, \bar{z})\mathrm{d}\bar{z} \left(\partial_w K_\epsilon(z, w)\right).$$

The factor n comes from the contraction of the tensors depending on V. Next, we compute

$$\partial_w K_\epsilon(z, w) = \frac{1}{4\pi\epsilon} \frac{\bar{z} - \bar{w}}{4\epsilon} e^{-|\bar{z} - \bar{w}|^2/4\epsilon}.$$

Similarly,

$$\partial_z P_\epsilon^L(z,w) = \int_{t=\epsilon}^L \frac{1}{4\pi t}\frac{(\bar{z}-\bar{w})^2}{(4t)^2} e^{-|\bar{z}-\bar{w}|^2/4t}\mathrm{d}t.$$

After making the change of coordinates $(y = z - w, w)$, and plugging in the expressions above we obtain an expression for the integral inside the $\epsilon \to 0$ limit

$$\frac{1}{16\pi^2}\int_{\mathbf{C}_y\times\mathbf{C}_w} fg\mathrm{d}^2y\mathrm{d}^2w \int_{t=\epsilon}^L \frac{1}{\epsilon t}\frac{1}{(4\epsilon)(4t)^2}\bar{y}^3 \exp\left(-\frac{1}{4}\left(\frac{1}{t}+\frac{1}{\epsilon}\right)|y|^2\right). \quad (9)$$

If φ is any compactly supported function then integration by parts yields the relation

$$\int_y \varphi(y)\bar{y}^k e^{-a|y|^2}\mathrm{d}^2y = \frac{1}{a^k}\int_y \left(\partial_y^3\varphi\right)(y)e^{-a|y|^2}\mathrm{d}^2y.$$

Applying this to the integral in (9), we obtain

$$\alpha[L](f\partial_z, g\mathrm{d}\bar{w}\partial_w) = n\lim_{\epsilon\to 0}\frac{1}{16\pi^2}\int_{\mathbf{C}_y\times\mathbf{C}_w}\partial_y^3(fg)(y,w)$$
$$\int_{t=\epsilon}^L \frac{\epsilon}{(\epsilon+t)^3}\exp\left(-\frac{1}{4}\left(\frac{1}{t}+\frac{1}{\epsilon}\right)\right).$$

Finally, performing integration in the y-direction using Wick's formula, the right-hand side becomes

$$n\frac{1}{2\pi}\left(\int_{\mathbf{C}_w}\partial_w^3 fg\mathrm{d}^2w\right)\lim_{\epsilon\to 0}\int_{t=\epsilon}^L \frac{\epsilon^2 t}{(\epsilon+t)^4}\mathrm{d}t.$$

The t-integral converges and in the $\epsilon \to 0$ limit

$$\int_{t=\epsilon}^L \frac{\epsilon^2 t}{(\epsilon+t)^4} \xrightarrow{\epsilon\to 0} \frac{1}{12}.$$

Note that there is no longer a dependence on the $L > 0$ parameter. This means that for any L the functional $\alpha = \alpha[L]$ is already a local functional representing the shifted central extension. In conclusion, we have calculated

$$\alpha(f\partial_z, g\mathrm{d}\bar{z}\partial_z) = \frac{1}{2\pi}\frac{n}{12}\int_{\mathbf{C}_z}\partial_z^3 fg\mathrm{d}^2z.$$

This is precisely the defining cocycle for the Virasoro factorization algebra of charge $c = n$. In conclusion, we see that the map of factorization algebras (8) becomes

$$\Phi : \mathcal{V}\mathrm{ir}_{c=n} \to \mathrm{Obs}_n^q$$

as desired.

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