Quantum Field Theory Seminar

Yang Mill's Talk

Christopher La Fond

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By popular request our motivating example throughout this talk will be classical electromagnetism in a vacuum, so we first begin with a quick refresher on the Maxwell-Field equations. From the physicists point of view, classical electromagnetism boils down to studying the following set of partial differential equations, known as the Maxwell-Field equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{E} = 0 \tag{1}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0 \tag{2}$$

With these equations, it can be shown that $\mathbf{E}$ and $\mathbf{B}$ can be described by a scalar function $V : \mathbb{R}^{1,3} \to \mathbb{R}$, and a vector field $\mathbf{M} : \mathbb{R}^{1,3} \to \mathbb{R}^3$ in the following way:

$$\begin{aligned}\mathbf{E} =& -\nabla V - \frac{\partial \mathbf{M}}{\partial t}\\ \mathbf{B} =& \nabla \times \mathbf{M}\end{aligned}$$

One then finds that the $\mathbf{E}$ and $\mathbf{B}$ are invariant under the following transformation in the potentials:

$$\begin{aligned}V' =& V - \frac{\partial \lambda}{\partial t}\\ \mathbf{M}' =& \mathbf{M} + \nabla \lambda\end{aligned}$$

for any function $\lambda : \mathbb{R}^{1,3} \to \mathbb{R}$. This transformation can be better encoded by defining the four potential on $\mathbb{R}^{1,3}$:

$$A^i \partial_i = (V, M_x, M_y, M_z)$$

We then obtain the following one form under the musical isomorphism:

$$A_i dx^i = -V dt + M_i dx^i$$

Then the transformation is given by:

$$A' = A + d\lambda$$

In physics, this is called a gauge transformation, and it was the goal of the original Yang-Mills paper to extend this invariance to the strong force. In so doing, they, perhaps by happenstance, implicitly used geometry and the theory of connections to develop the Yang-Mills equations.

We now turn to developing the ingredients behind this theory. Recall from the last talk, that given a principal G bundle P over some base manifold M, we can prescribe the principal bundle with a connection A that determines the horizontal subspace of TP. We can view A as a Lie algebra value one form on P satisfying:

$$A(X_p) = \begin{cases} X \in \mathfrak{g} & \text{if } X_p \in V_p \\ 0 & \text{if } X_p \in H_p \end{cases}$$

we call A a vertical one form on P. Furthermore, recall that the curvature of such a connection is given by:

$$F = \pi^* dA$$

where π is the projection $P \to M$. One can check by direct verification that this is equivalent to the structure equation:

$$F = dA + \frac{1}{2}[A, A]$$

where for Lie algebra valued k and l forms:

$$[\eta, \omega] = \sum_{i,j=0}^{n} \eta^i \wedge \omega^j \otimes [T_i, T_j]$$

after choosing a basis $\{T_i\}$ for $\mathfrak{g}$. For one forms this reduces to the formula:

$$[\eta, \omega](X, Y) = [\eta(X), \omega(Y)] - [\eta(Y), \omega(X)]$$

Recall that in a local trivialization of P, corresponding to local section $s : U \to P$, we have that:

$$A_s = s^* A$$

which is then a Lie algebra valued on form on the base manifold M. Furthermore, we have that $F_s = s^* F$, satisfies a similar structure equation:

$$F_s = dA_s + \frac{1}{2}[A_s, A_s]$$

Now let $U_i \times G$ and $U_j \times G$ be two local trivializations of P corresponding to sections s_i and s_j such that $U_i \cap U_j \neq \emptyset$. Then, we have that on the overlap :

$$s_j = s_i \cdot g_{ij}$$

for some $g_{ij} : U_i \cap U_j \to G$, which we call a local gauge transformation. Then under a local gauge transformation we have that:

$$A_j = \mathrm{Ad}_{g_{ij}^{-1}} \circ A_i + g_{ij}^* \theta$$

where θ is the Maurer-Cartan form:

$$\theta(v) = D_g L_{g^{-1}}(v)$$

The above can be verified quite easily by direct calculation with pushforwards, and using the fact that for:

$$\begin{aligned} \Phi : M \times G &\longrightarrow M \\ : (x, g) &\longmapsto x \cdot g \end{aligned}$$

we have for $(X, Y) \in T_x M \oplus T_g G$:

$$D_{(x,g)} \Phi(X, Y) = D_x R_g(X) + D_g \phi_x(Y)$$

where ϕ_x is the orbit map through the point x. In a similar vein, if A is the global connection one form on P, and f is a global bundle automorphism, then we can write that:

$$f(p) = p \cdot \sigma_f(p)$$

for some $\sigma_f : P \to G$ satisfying:

$$\sigma(p \cdot g) = g^{-1} \sigma_f(p) g$$

Then A transforms as:

$$f^* A = \mathrm{Ad}_{\sigma_f^{-1}} \circ A + \sigma_f^* \theta$$

Furthermore, F_{s_i} transforms as:

$$F_{s_j} = \mathrm{Ad}_{g_{ij}^{-1}} \circ F_{s_i}$$

We then see that similarly:

$$f^* F = \mathrm{Ad}_{\sigma^{-1}} \circ F$$

Turning back to the physics for a moment, we make the assumption that classical electromagnetism corresponds to a $U(1)$ gauge theory over the base manifold, then:

$$P = \mathbb{R}^{1,3} \times U(1)$$

so there exists a global section of P. We can then view A_s as a global connection one form on the base with values in $\mathfrak{u}(1) \cong \mathbb{R}$, and set:

$$A_{s_i} = -Vdt + M_x dx + M_y dy + M_z dz$$

then under a the change of section $g_{ij} = e^{i\lambda(x)}$ we have:

$$\begin{aligned} A_{s_j} =& \mathrm{Ad}_{g_{ij}^{-1}} \circ A_{s_i} + g_{ij}^{-1} dg_{ij} \\ =& A_{s_i} + e^{-i\lambda(x)} de^{i\lambda(x)} \\ =& A_{s_i} + d\lambda \end{aligned}$$

thus we obtain the gauge transformation for the four potential discussed earlier. Furthermore, as $U(1)$ is abelian we have that:

$$F_{s_i} = dA_s$$

and while I won't do the full calculation out, I will examine an easy term:

$$\begin{aligned} F_{xt} =& -\frac{\partial V}{\partial x} dx \wedge dt - \frac{\partial M_x}{\partial t} dx \wedge dt \\ =& E_x dx \wedge dt \end{aligned}$$

so the curvature form is given in matrix notation as:

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

so the curvature form contains the information for both the electric and the magnetic fields. As $U(1)$ is abelian, we have that $\mathrm{Ad}_{g^{-1}}$ is the identity, so F_s, and as a consequence the physical fields, are invariant under a gauge transformation.

Recall that a vector bundle E associated to P, with fibre V, and a representation ρ is defined as:

$$E = P \times_\rho V$$

which is the quotient of $P \times V$ under the equivalence relation:

$$[p, v] \sim [p \cdot g, \rho(g)^{-1} \cdot p]$$

In particular, this implies that:

$$[p \cdot g, v] \sim [(p \cdot g) \cdot g^{-1}, \rho(g)v] \sim [p, \rho(g)v]$$

If the representation is trivial we have that E is trivial, hence:

$$E = M \times V$$

A local section Φ of the bundle is then the equivalence class:

$$\Phi = [s(x), \phi(x)]$$

where $\phi : U \subset M \to V$, and $s : U \to P$. If f is a global bundle automorphism, we have that the action of f on an associated vector bundle is given by:

$$f \cdot [p, v] = [f(p), v] = [p \cdot \sigma_f(p), v]$$

The connection one form induces a a covariant derivative on E:

$$\begin{aligned} \nabla^A : \Gamma(E) &\longrightarrow \Omega^1(M, E) \\ \Phi &\longmapsto [p, d\phi + \rho_*(A_s)\phi] \end{aligned}$$

We can also define the covariant exterior derivative:

$$d_A : \Omega^k(M, E) \longrightarrow \Omega^{k+1}(M, E)$$

In a local frame $\{e_i\}$ of E we see that $\omega \in \Omega^k(M, E)$ can be written as:

$$\omega = \omega^i \otimes e_i$$

for $\omega^i \in \Omega^k(M)$, then:

$$d_A\omega = d\omega^i \otimes e_i + (-1)^k \omega \otimes \nabla^A e_i$$

In a local gauge $s : U \to P$, we choose a basis for $v_1, \ldots, v_n$ for V, then determine a local frame $e_i, \ldots e_n$ for E via:

$$e_i = [s, v_i]$$

Then:

$$\omega_s = \omega^i \otimes v_i$$

we can then write that:

$$(d\omega)_s = d\omega_s + \rho_*(A_s) v_i \wedge \omega_s^i \overset{\text{def}}{=} d\omega_s + A_s \wedge \omega_s$$

An associated vector bundle of particular interest is $\mathrm{Ad}(P)$:

$$\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$$

Why is this vector bundle important? Recall that F is horizontal, that is, it sends every vertical vector field to zero. We denote general k forms with values in $\mathfrak{g}$, which transform like the curvature form, and are horizontal by $\Omega^k_{\text{Hor}}(P,\mathfrak{g})^{\text{Ad}}$. Note that every F is in $\Omega^2_{\text{Hor}}(P,\mathfrak{g})^{\text{Ad}}$, and that given a connection $A \in \Omega(P,\mathfrak{g})$, any other connection can be written as:

$$A' = A + \omega$$

for an $\omega \in \Omega^1_{\text{Hor}}(P,\mathfrak{g})^{\text{Ad}}$. It turns out that the vector space $\Omega^k_{\text{Hor}}(P,\mathfrak{g})$ is *canonically* isomorphic to the vector space $\Omega^k(M, \text{Ad}(P))$, via the map:

$$\Lambda : \Omega^k_{\text{Hor}}(P,\mathfrak{g}) \longrightarrow \Omega^k(M, \text{Ad}(P))$$

defined by:

$$\Lambda(\omega)(X_1, \cdots, X_k) = [p, \omega(Y_1, \ldots, Y_k)] \in \text{Ad}(P)_x$$

where:

$$\pi(p) = x \qquad \text{and} \qquad \pi_*(Y_i) = X_i$$

Verification of the above statement is overall pretty standard, one checks that the map is well defined, linear, and bijective, and the statement follows. To see this in the $k = 0$ case note that:

$$\Omega^0_{\text{Hor}}(P,\mathfrak{g})^{\text{Ad}} = \{f \in C^\infty(P,\mathfrak{g}) : f(p \cdot g) = \text{Ad}_{g^{-1}} \circ f(p)\}$$

while:

$$\Omega^0(M, \text{Ad}(P)) = \Gamma(\text{Ad}(P))$$

Looking at the equivalence class:

$$[p, f(p)]$$

we see that:

$$[p \cdot g, f(p \cdot g)] = [p \cdot g, \text{Ad}_{g^{-1}} \circ f(p)] = [p, f(p)]$$

so each element in $\Omega^0_{\text{Hor}}(P,\mathfrak{g})$ completely determines a global section of $\text{Ad}(P)$ and vice versa. This then implies the following statements:

(i) The set of all connection on P is an affine space over $\Omega^1(M, \text{Ad}(P))$

(ii) The curvature F^A of a connection A on P can be identified with an element $F^A_M \in \Omega^2(M, \text{Ad}(P))$. So local curvature forms on M, extend globally to 2-forms on M with values in $\text{Ad}(P)$

both of which are vital for Yang-Mills. Importantly, this implies we can write the Bianchi identity, = as:

$$d_A F^A_M = 0$$

We notice this by stating the Bianchi Identity:

$$dF^A + [A, F^A] = 0$$

which follows from the properties of the bracket operation on forms, and noting that in a local gauge, for $E = \mathrm{Ad}(P)$ we have that:

$$\begin{aligned} d_A\omega =&(d\omega_s) + \rho_*(A_s)T_i \wedge \omega^i \\ =&(d\omega_s) + A^j_s \wedge \omega^i \otimes [T_j, T_i] \\ =&(d\omega_s) + [A_s, \omega_s] \end{aligned}$$

So we have that since F_s can be extended to F^A_M:

$$dF^A + [A, F^A] = 0 \Rightarrow dF^A_M + [F^A_M, F^A_M] = d_A F^A_M = 0$$

We are now going to quickly define the other necessary ingredients in the Yang-Mills Lagrangian. Suppose that M has a (pseudo)-Riemannian metric g, and recall that we can raise the indices of k form ω via:

$$\omega^{i_1\cdots i_k} = g^{i_1 j_1} \cdots g^{i_k j_k} \omega_{j_1\cdots j_k}$$

Via this operation we define the scalar product of forms by:

$$\langle \omega, \eta \rangle = \frac{1}{k!} \omega_{i_1\cdots i_k} \eta^{i_1\cdots i_k}$$

For twisted forms, i.e. forms with values in some vector bundle E, if E carries a bundle metric $\langle \cdot, \cdot \rangle_E$, we can define a the scalar product of twisted forms by:

$$\langle \omega, \eta \rangle = \langle \omega^i, \eta^j \rangle \langle e_i, e_j \rangle_E$$

The L_2 norm for k-forms is:

$$\langle \omega, \eta \rangle = \int_M \langle \omega, \eta \rangle \mathrm{dvol}_g$$

While the L_2 product for twisted k-forms is:

$$\langle \omega, \eta \rangle_{E, L^2} = \int_M \langle \omega, \eta \rangle_E \mathrm{dvol}_g$$

Furthermore, we define the hodge star operator as the unique linear map:

$$\star : \Omega^k(M) \to \Omega^{n-k}(M)$$

defined by:

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle \mathrm{dvol}_g$$

In a local oriented orthonormal frame we have that:

$$\star(\alpha^{m_1} \wedge \cdots \wedge \alpha^{m_k}) = g^{m_1 m_1} \cdots g^{m_k m_k} \epsilon_{m_1 \cdots m_k m_{k+1} \cdots m_n} \alpha^{m_{k+1}} \wedge \cdots \wedge \alpha^{m_n}$$

where $\{m_1, \ldots, m_k\}$ is complimentary to the set $\{m_{k+1}, \ldots, m_n\}$ and ϵ is totally anti-symmetric with:

$$\epsilon_{123\cdots n} = 1$$

In particular:

$$\star\mathrm{dvol}_g = (-1)^t \qquad \text{and} \qquad \star 1 = \mathrm{dvol}_g$$

The hodge star on twisted forms just ignores the tensored section of E, i.e.:

$$\star\omega = (\star\omega^i) \otimes e_i$$

We further define the codifferential, $d^\star : \Omega^k(M) \to \Omega^{k-1}$ as:

$$d^\star = (-1)^{t+nk+1} \star d \star$$

and the covariant codifferential as:

$$d_A^\star = (-1)^{t+nk} \star d_A \star$$

We can now finally construct the Yang-Mills Lagrangian. We fix the following data:

- an n-dimensional pseudo-Riemannian manifold (M, g)
- a principle G bundle $P \to M$ with compact structure group G
- an Ad-invariant positive definite scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$.

The Ad invariant scalar product determines a bundle metric on the associated real vector bundle $\mathrm{Ad}(P)$ that we denote by $\langle \cdot, \cdot \rangle_{\mathrm{Ad}(P)}$ via:

$$\langle [p, v], [p, w] \rangle_{\mathrm{Ad}}(p) = \langle v, w \rangle_{\mathfrak{g}}$$

The Yang-Mills Lagrangian is then given by:

$$\mathscr{L}_{YM}[A] = -\frac{1}{2} \left\langle F^A_M, F^A_M \right\rangle_{\mathrm{Ad}(P)}$$

Note that this Lagrangian is gauge invariant, as:

$$F^{f^*A} = f^* F^A$$

for some global bundle automorphism f. We can rewrite this as:

$$f^*F^A = R_{\sigma_f}^* F^A = \mathrm{Ad}_{\sigma_f^{-1}} \circ F^A$$

Hence, with our previous definition of the action of a global bundle automorphism on an associated vector bundle, we have that:

$$F_M^{f^*A} = f^{-1} \cdot F_M^A$$

F_M^A takes values in $\mathrm{Ad}(P)$, so we have that f^{-1} acts on F_M^A via the adjoint action, and since the scalar product on $\mathrm{Ad}(P)$ is Ad invariant, we have that:

$$\left\langle F_M^A, F_M^A \right\rangle_{\mathrm{Ad}(P)} = \left\langle f^{-1} \cdot F_M^A, f^{-1} \cdot F_M^A \right\rangle_{\mathrm{Ad}(P)}$$

hence:

$$\mathscr{L}_{YM}[f^*A] = \mathscr{L}_{YM}[A]$$

so the Lagrangian is gauge invariant as desired. The Yang-Mills action is given by:

$$\mathscr{S}_{YM}[A] = -\frac{1}{2}\int_M \left\langle F_M^A, F_M^A \right\rangle_{\mathrm{Ad}(P)} \mathrm{dvol}_g$$

A connection A is a critical point of the Yang-Mills action if for all $\alpha \in \Omega^1_{\mathrm{Hor}}(P, \mathfrak{g})^{\mathrm{Ad}}$:

$$\left.\frac{d}{dt}\right|_{t=0} \mathscr{S}_{YM}[A + t\alpha] = 0$$

We will now calculate the critical points of $\mathscr{S}_{YM}$ as follows. First note that:

$$\begin{aligned} F^{A+t\alpha} &= dA + \frac{1}{2}[A, A] + td\alpha + t[A, \alpha] + t^2[\alpha, \alpha] \\ &= F^A + td\alpha + t[A, \alpha] + t^2[\alpha, \alpha] \end{aligned}$$

Both F and α have representatives in $\Omega^k(M, \mathrm{Ad}(P))$, so, by our earlier work with the Bianchi identity we have:

$$F_M^{A+t\alpha_M} = F_M^A + td_A\alpha_M + t^2[\alpha_M, \alpha_M]$$

So:

$$\left\langle F_M^{A+t\alpha_M}, F_M^{A+t\alpha_M} \right\rangle_{\mathrm{Ad}(P)} = \left\langle F_M^A, F_M^A \right\rangle_{\mathrm{Ad}(P)} + 2t\left\langle d_A\alpha_M, F_M^A \right\rangle_{\mathrm{Ad}(P)} + \mathscr{O}(t^2)$$

Hence:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathscr{S}_{YM}[A + t\alpha] &= 2\left\langle d_A\alpha_M, F_M^A \right\rangle_{\mathrm{Ad}(P), L^2} \\ &= 2\left\langle \alpha_M, d^\star F_M^A \right\rangle_{\mathrm{Ad}(P), L^2} \end{aligned}$$

Since this needs to hold for all α_M, and since the L^2 norm is nondegenerate, we obtain the Yang-Mills equation:

$$d_A^\star F = 0 \Rightarrow d_A \star F = 0$$

The Yang-Mills equation, combined with the Bianchi-Identity are Maxwell's equations in a vacuum. Let us briefly see this, Recall that:

$$F_{xt} = E_x dx \wedge dt \Rightarrow \star F_{xt} = E_x dy \wedge dz$$

Note that d_A reduces to d as the representation of $U(1)$ on $\mathrm{Ad}(P)$ is trivial, or more concretely, since we have identified $\mathfrak{u}(1)$ with $\mathbb{R}$, and since $\mathrm{Ad}(P)$ of is trivial, we can think of F as a regular 2 form on $\mathbb{R}^{1,3}$. So we obtain:

$$d \star F_{xt} = \frac{\partial E_x}{\partial x} dx \wedge dy \wedge dz$$

Similarly, the purely spacial components the other components of are $\mathbf{E}$:

$$d \star F_{yt} = \frac{\partial E_y}{\partial y} dx \wedge dy \wedge dz$$
$$d \star F_{zt} = \frac{\partial E_z}{\partial z} dx \wedge dy \wedge dz$$

Hence we see that:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \Rightarrow \nabla \cdot \mathbf{E} = 0$$

We have recovered one of Maxwell's equations!

We now turn to some complex geometry; let Σ be a Riemann surface. In particular this implies that Σ is orientable, has a complex structure, and carries a Reimannian metric. The Hodge star operator then sends one forms to one forms, and satisfies:

$$\star^2 = -1$$

Importantly, via the musical isomorphism $T^*\Sigma \leftrightarrow T\Sigma$ it turns out that any orientable 2 manifold carries an almost complex structure, and any 2 manifold with an almost complex structure is orientable. The hodge star on Σ determines a splitting:

$$\Omega^1(\Sigma) \otimes \mathbb{C} \cong \Omega^{(1,0)}(\Sigma) \oplus \Omega^{(0,1)}(\Sigma)$$

corresponding to the eigenvalues i and $-i$ of $\star$. We then have that the deRham differential splits into:

$$d = \partial + \bar{\partial}$$

where:

$$\partial : \Omega^{(k,0)} \to \Omega^{(k+1,0)}$$
$$\bar{\partial} : \Omega^{(0,k)} \to \Omega^{(0,k+1)}$$

Thought, this does not hold in general, it is true for silly dimensional reason that:

$$\bar{\partial}^2 = 0$$

because $\Omega^{(0,2)}(\Sigma) = \Omega^{(2,0)}(\Sigma) = 0$. If P is a principal bundle over Σ, and we have a connection A, then we have a similar splitting of the exterior covariant derivative on $\mathrm{Ad}(P)$ i.e.:

$$d_A = \left(\partial + A_s^{(1,0)}\right) + \left(\bar{\partial} + A_s^{(0,1)}\right) = \partial_A + \bar{\partial}_A$$

We note that $\mathrm{Ad}(P)$ is a holomorphic as:

$$\bar{\partial}_A^2 = 0$$

for the same dimension reasons as before, and that holomorphic sections of $\mathrm{Ad}(P)$ are ones such that:

$$\bar{\partial} X = 0$$

The Yang-Mills equations then states that:

$$d_A \star F = \partial_A \star F + \bar{\partial}_A \star F = 0$$

We need both do be zero, as they are independent sections of $\Omega^1(\Sigma)$, hence we see that the Yang-Mills equations reduces to the statement:

$\star F$ is a holomorphic section of $\mathrm{Ad}(P)$ that is covariant constant