Koszul Duality for E_n Algebras

Yuchen Fu

October 2, 2018

This is the note for a talk given at Joint Northeastern–MIT Graduate Research Seminar "The Yangian and Four-dimensional Gauge Theory" during Fall 2018.

Assumptions We work over a field k of characteristic 0 and implicitly work in the $(\infty,1)$-setting; so by the word "category" we always mean an $(\infty,1)$-category. The background "category of categories" is the category of stable presentable $(\infty,1)$-categories with only colimit-preserving morphisms between them. This category is equipped with a symmetric monoidal structure given by the Lurie tensor product. The category Vect is the $(\infty,1)$-category of (unbounded) chain complexes over k.

1 Operads, Algebras and Modules

1.1 Operads

We shall use the following version of definition from [FG12]. Let $\mathcal{X}$ be a symmetric monoidal category. Let Σ be the category of (nonempty) finite sets and bijections. Let $\mathcal{X}^\Sigma := \prod_{n\geq 1} \mathrm{Rep}_{\mathcal{X}}(\Sigma_n)$ be the category of symmetric sequences in $\mathcal{X}$; its objects are collections $\{O(n) \in \mathcal{X}, n \geq 1\}$ such that Σ_n acts on $O(n)$. Observe that $\mathcal{X}^\Sigma \simeq \mathrm{Funct}(\Sigma, \mathcal{X})$. This category admits a monoidal structure $\star$ such that the following functor is monoidal:

$$\mathcal{X}^\Sigma \to \mathrm{Funct}(\mathcal{X}, \mathcal{X})$$

$$\{O(n)\} \mapsto \left(x \mapsto \bigoplus_{n\geq 1} (O(n) \otimes x^{\otimes n})_{\Sigma_n} \right)$$

Namely it's given by

$$P \star Q = \bigoplus_{n\geq 1} (P(n) \otimes Q^{\odot n})_{\Sigma_n}$$

where $\odot$ is the Day convolution:

$$(P \odot Q)(n) = \bigoplus_{i+j=n} \mathrm{Ind}_{S_i\times S_j}^{S_n} (P(i) \otimes Q(j))$$

Note that Day convolution is symmetric monoidal because $S_i \times S_j$ and $S_j \times S_i$ are conjugate in S_n. The unit object of $(\mathcal{X}^\Sigma, \star)$ is $\mathbf{1}_\star$, given by $\mathbf{1}_\star(1) = \mathbf{1}_{\mathcal{X}}$ and $\mathbf{1}_\star(n) = 0_{\mathcal{X}}\ \forall n > 1$.

We define $\mathrm{Oprd}(\mathcal{X})$, the category of *reduced, augmented* operads over $\mathcal{X}$, to be that of augmented associative algebras in $(\mathcal{X}^\Sigma, \star)$ for which $\mathbf{1}_{\mathcal{X}} \to \mathcal{O}(1)$ is an isomorphism. The dual notion $\mathrm{coOprd}(\mathcal{X})$ of co-augmented cooperads is defined dually. This means that we have the composition maps

$$O(k) \otimes O(n_1) \otimes \ldots \otimes O(n_k) \to O(n_1 + \ldots + n_k)$$

as well as a unit element in $O(1)$, such that the unital, associative and equivariance laws are satisfied up to coherent homotopy. If we interpret the definition in the classical (non-∞) setting, then we obtain the usual notion of operads.

Example 1. *The associative operad Ass is given by that $Ass(n) = k[\Sigma_n]$, the regular representation of Σ_n; the operad maps come from substitution. Similarly, the commutative operad Comm is given by $Comm(n) = k$, the trivial representation of Σ_n.*

Linear Dual of Operads Given an operad O such that $O(n)$ has finite dimensional cohomologies, we can define O^* to be $O^*(n) = O(n)^*$, which will be a cooperad.

Shifting Operads For an operad $O \in \mathrm{Oprd}(\mathrm{Vect})$, we use $O[1]$ to denote the operad given on the component level by

$$O[1](n) = O\widetilde{(n)[n}-1]$$

(where the tilde indicates that the Σ_n action needs to be twisted accordingly), such that $c \mapsto c[1]$ gives an equivalence $O[1]$-alg(Vect) $\to O$-alg(Vect) (see below). The dual notion of suspension of cooperads is defined analogously; namely, we also require $O[1]$-coalg(Vect) $\to O$-coalg(Vect) is given by $c \mapsto c[1]$.

1.2 Algebras over Operads

Let $\mathcal{X}$ be as before, and let $\mathcal{C}$ be a commutative algebra object in the category of $\mathcal{X}$-modules. The action

$$(O, c) \mapsto \bigoplus_n (O(n) \otimes c^{\otimes n})_{\Sigma_n}$$

defines the $\star$-action of $\mathcal{X}^\Sigma$ on $\mathcal{C}$. For any operad O and any cooperad O°, define

$$O\text{-alg}(\mathcal{C}) := O\text{-mod}(\mathcal{C}, \star)$$

to be the category of O-algebras in $\mathcal{C}$ and

$$O^\circ\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}}(\mathcal{C}) := O^\circ\text{-comod}(\mathcal{C}, \star).$$

to be the category of O°-coalgebras in $\mathcal{C}$.

Example Algebras over Ass and Comm in a category $\mathcal{C}$ correspond respectively to augmented associative (that is, A_∞) and augmented commutative (that is, E_∞) algebras in $\mathcal{C}$; Similarly, coalgebras over Ass* and Comm* in $\mathcal{C}$ correspond to coaugmented coassociative coalgebras and coaugmented cocommutative coalgebras in $\mathcal{C}$.

Remark 1. *Strictly speaking, the augmentation does not come from being a module of the operad, but rather the obvious equivalence of categories $Assoc^{non\text{-}unital}(\mathcal{C}) \simeq Assoc^{aug}(\mathcal{C})$, given by direct sum with **1** / taking the augmentation ideal. To simplify discussion, we'll consider associative algebras as augmented for the rest of this talk.*

1.2.1 Four Types of Comodules

Notice that what we wrote was O°-coalg$^{\mathrm{nil}}_{\mathrm{d.p.}}(\mathcal{C})$ and not O°-coalg; indeed the former doesn't in general specialize to what we usual call comodules of cooperads. (Observe that, if A is a coalgebra, we ought to have maps $A \to A^{\otimes n}$ and therefore a map to the direct *product*.) Instead, define the following $*$-action:

$$(O, c) \mapsto \prod_n (O(n) \otimes c^{\otimes n})^{\Sigma_n}$$

and write

$$O^\circ\text{-coalg}(\mathcal{C}) := O^\circ\text{-comod}(\mathcal{C}, *)$$

Then this is the one that specializes to our usual notion.

In addition, define the category O-coalg$^{\mathrm{nil}}$ to be the one equipped with the action

$$(O, c) \mapsto \bigoplus_n (O(n) \otimes c^{\otimes n})^{\Sigma_n}$$

and O-coalg$_{\text{d.p.}}$ the one equipped with the action

$$(O, c) \mapsto \prod_n (O(n) \otimes c^{\otimes n})_{\Sigma_n}.$$

For this talk we will not worry about the d.p. part, since we have the averaging functor

$$\text{avg} : O^\circ\text{-coalg}^{\text{nil}}_{\text{d.p.}} \to O^\circ\text{-coalg}^{\text{nil}}, \text{avg} : O^\circ\text{-coalg}_{\text{d.p.}} \to O^\circ\text{-coalg}$$

which is an isomorphism in characteristic 0. We also have the obvious functor

$$O^\circ\text{-coalg}^{\text{nil}} \to O^\circ\text{-coalg}$$

We compose those two to get a map

$$\text{res} : O^\circ\text{-coalg}^{\text{nil}}_{\text{d.p.}} \to O^\circ\text{-coalg}.$$

This functor commutes with colimits so admit a right adjoint, giving a pair

$$\text{res} : O^\circ\text{-coalg}^{\text{nil}}_{\text{d.p.}} \rightleftarrows O^\circ\text{-coalg} : \text{res}^R$$

Conjecture 1 ([FG12]). *res is always fully faithful.*

For categories of a specific type, this complication (and many below) disappears; namely those that are pro-nilpotent:

Definition 1. *A category C is called pro-nilpotent if we can write it as $C = \lim_{\mathbb{N}^{op}} C_i$ in the category of stable symmetric monoidal $\mathcal{X}$-module categories, such that the following are satisfied:*

1. $C_0 \simeq 0$;
2. $i \geq j \implies f_{i,j} : C_i \to C_j$ *commutes with limits;*
3. *The monoidal map $C_i \otimes C_i \to C_i$, when restricted to $\ker f_{i,i-1} \otimes C_i$, is zero.*

Example 2. *The category $\mathcal{X}^\Sigma$ is pro-nilpotent. Namely, C_i is the full subcategory of those sequences whose value on $n \geq i$ is 0.*

Remark 2. *For the results in [Cos13], the base category is that of chain complexes over a complete filtered vector space, such that each graded piece is a bounded complex. By truncating on the filtration, we can see that this category is pro-nilpotent, so all "nice" results below apply.*

Remark 3. *One of the bootstrapping observations of [FG12] is that $D(\mathit{Ran}\ X)$, equipped with the chiral tensor structure, is pro-nilpotent. Namely, the strata come from considering Ran $X^{\leq n}$, which is given by the same construction as Ran space, but only gluing along $\Delta : X^I \to X^J$ when $|J| \leq n$. Note that $D(\mathit{Ran}\ X)$ equipped with the $*$-tensor structure is not pro-nilpotent.*

Proposition 1 ([FG12]). *When $\mathcal{C}$ is pro-nilpotent, res is an isomorphism.*

1.3 Modules

Let $A \in \mathcal{C}$ be an O-algebra, and let $\mathcal{M}$ be a module category over $\mathcal{C}$ in the category of $\mathcal{X}$-modules. Note that there is a symmetric monoidal category $\text{Sqz}(\mathcal{C}, \mathcal{M})$, the "square zero extension" of $\mathcal{C}$ by $\mathcal{M}$, obtained from $\mathcal{C} \times \mathcal{M}$ by collapsing the $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ morphisms. We then define the category of A-modules in $\mathcal{M}$, denoted $\text{Mod}_A(\mathcal{M})$, to be the $((\infty, 1)$-categoric) fiber of A under $\pi_1 : O\text{-alg}(\text{Sqz}(\mathcal{C}, \mathcal{M})) \to O\text{-alg}(\mathcal{C})$, which is induced by the projection $\pi_1 : \mathcal{C} \times \mathcal{M} \to \mathcal{C}$. The dually defined comodule category is denoted $\text{Comod}^{\text{nil}}_{A^!,\text{d.p.}}(\mathcal{M})$.

Concretely speaking, an A-module structure amounts to an object $M \in \mathcal{M}$ and operation maps

$$O(n) \otimes A^{k-1} \otimes M \otimes A^{n-k} \to M$$

for each $1 \leq k \leq n$, such that all necessary conditions hold.

Left/Right Module For the operad Ass, the notion above recovers the notion of *bimodules* over an associative algebra A. Using colored operads it is also possible to recover the notion of left/right modules, as is detailed in [Lur]. We shall not define those concepts, but for the sake of stating results let us introduce the notation $\mathrm{LMod}_A(\mathcal{M})$ and $\mathrm{RMod}_A(\mathcal{M})$ to denote those two categories.

2 E_n operads

For this talk we shall focus on the case of E_n operads. Namely, for each $n \geq 1$, there is an element $\mathcal{E}_n \in \mathrm{Oprd}(\mathrm{Spc})$ that is realized by the little n-disk or the little n-cube operads. The operad in Oprd(Vect) induced by the singular chain functor $C_* : \mathrm{Spc} \to \mathrm{Vect}$ is then called the E_n operad in chain complexes; we will refer to it simply by E_n.

By definition we have $E_1 \simeq \mathrm{Ass}$, so an E_1-algebra is nothing more than an augmented associative algebra. The other extreme is when $n = \infty$, for which we'll write $E_\infty := \mathrm{colim}_n E_n$. It turns out $E_\infty \simeq \mathrm{Comm}$ (having to do with S^∞ being contractible), i.e. E_∞-algebras are augmented commutative algebras. The other E_n cases are interpolations between those two, so can be seen as describing algebras that are "partially commutative". More precisely, there is a sequence of maps between operads

$$E_1 \to E_2 \to \ldots \to E_n \to E_{n+1} \to \ldots \to E_\infty$$

induced from the topological counterpart

$$\mathcal{E}_1 \to \mathcal{E}_2 \to \ldots \to \mathcal{E}_n \to \mathcal{E}_{n+1} \to \ldots \to \mathcal{E}_\infty$$

(where $\mathcal{E}_\infty(n) = *$ for each n) by the standard embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$.

From now on, Vect will denote the homotopy category of chain complexes. When the category $\mathcal{C}$ is not specified, by E_n-algebras we mean elements of E_n-alg(Vect).

3 Koszul Duality

3.1 Bar Construction for Associative Algebras

Let $\mathcal{A}$ be a monoidal $\mathcal{X}$-module category with limits and colimits, then we have a standard construction of a pair of adjoint functors

$$\mathrm{Bar} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathcal{A}) \rightleftarrows \mathrm{CoassocCoalg}^{\mathrm{coaug}}(\mathcal{A}) : \mathrm{coBar}$$

where Bar maps R to $\mathbf{1} \otimes_R \mathbf{1}$ ($\mathbf{1}$ is considered as both a left and a right R-module, by means of the augmentation), and coBar defined dually. The comultiplication on $\mathrm{Bar}(R)$ is given by the following:

$$\mathbf{1} \otimes_R \mathbf{1} \simeq \mathbf{1} \otimes_R R \otimes_R \mathbf{1} \to \mathbf{1} \otimes_R \mathbf{1} \otimes_R \mathbf{1} \simeq \mathbf{1} \otimes_R \mathbf{1} \otimes_{\mathbf{1}} \mathbf{1} \otimes_R \mathbf{1} \to (\mathbf{1} \otimes_R \mathbf{1}) \otimes (\mathbf{1} \otimes_R \mathbf{1})$$

The coaugmentation is given by

$$\mathbf{1} \simeq \mathbf{1} \otimes \mathbf{1} \to \mathbf{1} \otimes_R \mathbf{1}$$

It is checked in e.g. [Lur, Theorem 5.2.2.17] that this indeed lands in coassociative algebras.

Now let $\mathcal{A}$ be as above and let $\mathcal{C}$ be an $\mathcal{A}$-module category. Fix some augmented associative algebra $A \in \mathcal{A}$. By general construction we have an adjoint pair

$$\mathrm{Bar}_A : A\text{-mod}(\mathcal{C}) \rightleftarrows \mathcal{C} : \mathrm{triv}_A$$

Namely we have $\mathrm{Bar}_A(M) = M \otimes_A \mathbf{1}$ where $A \to \mathbf{1}$ is the augmentation; by this notation we mean the colimit of the following diagram:

$$\cdots A \otimes M \rightrightarrows M$$

Similarly if A° is an coaugmented coassociative coalgebra and $\mathcal{C}^\circ$ an $\mathcal{A}^\circ$-comodule, then we have an adjoint pair

$$\mathrm{triv}_{A^\circ} : \mathcal{C}^\circ \rightleftarrows A^\circ\text{-mod}(\mathcal{C}^\circ) : \mathrm{coBar}_{A^\circ}$$

3.2 Koszul Duality for Operads

When $\mathcal{A}$ is $\mathcal{X}^\Sigma$ as above, these functors trivially lift to another adjoint pair

$$\mathrm{Bar} : \mathrm{Oprd}(\mathcal{X}) \rightleftarrows \mathrm{coOprd}(\mathcal{X}) : \mathrm{coBar}$$

that are compatible with the obvious forgetful functors. This pair we call the operadic Koszul duality.

Proposition 2. *These are mutual equivalences.*

Proof. Apply the algebraic Koszul duality (defined below) on $\mathcal{X} = \mathrm{Vect}$ and $\mathcal{C} = \mathrm{Vect}^\Sigma$ this reduces to the computation that $\mathrm{Ass}^! = \mathrm{Ass}^*[1]$, which is done manually. □

We shall refer to $\mathrm{Bar}(x)$ as the *Koszul dual* of x and write it as $x^!$.

Example 3. *The fundamental example in representation theory is* $Lie^! = Comm^*[1]$*, corresponding to the relationship between an Lie algebra and its Chevalley complex.*

3.2.1 Koszul Dual for the E_n Operads

Proposition 3. $E_1^! = E_1^*[1]$*. More generally, we have* $E_n^! \simeq E_n^*[n]$*, and the map is compatible with* $E_n \to E_{n+1}$*.*

The $n = 1$ case is a straightforward computation. For our setting (characteristic 0) the general n case would follow from a corresponding computation in homology operad in [GJ94] plus the formality theorem for E_n proved in *loc. cit*; over $\mathbb{Z}$ this is proven in [Fre11].

3.3 Koszul Duality for Algebras

Now let $\mathcal{C}$ be the same as in section 1.2. For any Koszul pair $(O, O^!)$, bar construction for modules gives an adjoint pair:

$$\mathrm{Bar}_O^{\mathrm{naive}} : O\text{-alg}(\mathcal{C}) \rightleftarrows O^!\text{-coalg}^{\mathrm{nil}}_{\mathrm{d.p.}}(\mathcal{C}) : \mathrm{coBar}^{\mathrm{naive}}_{O^!}$$

Again compatible with the forgetful functors. Now combine with the restriction adjoint pair to get

$$\mathrm{Bar}_O = \mathrm{res} \circ \mathrm{Bar}_O^{\mathrm{naive}} : O\text{-alg}(\mathcal{C}) \rightleftarrows O^!\text{-coalg}(\mathcal{C}) : \mathrm{coBar}^{\mathrm{naive}}_{O^!} \circ \mathrm{res}^R = \mathrm{cobar}_{O^!}$$

This is what we call the algebraic Koszul duality, and we'll write $A^!$ for $\mathrm{Bar}_O(A)$ as well. We shall say A is *Koszul* if $A \to (A^!)^!$ is an isomorphism. Recall that when $\mathcal{C}$ is pro-nilpotent, the two functors agree.

Proposition 4 ([FG12, Prop 4.1.2])**.** *When* $\mathcal{C}$ *is pro-nilpotent, both functors are equivalences.*

The Two Bar Constructions Agree In the case $O = \mathrm{Ass}$, the Koszul duality above gives a pair of adjunction

$$[1] \circ \mathrm{Bar}_{\mathrm{Ass}} : \mathrm{Assoc}^{\mathrm{aug}}(\mathcal{C}) \rightleftarrows \mathrm{Coassoc}^{\mathrm{coaug}}(\mathcal{C}) : \mathrm{coBar}_{\mathrm{Ass}^*[1]} \circ [-1]$$

This agrees with the bar construction given at the beginning of section 3.

Example 4. *Taking Koszul dual along* $Ass^! = Ass^*[1]$ *gives Hochschild complex; along* $Lie^! = Comm^*[1]$ *gives Chevalley complex; and along* $Comm^! = Lie^*[1]$ *gives Harrison complex.*

3.3.1 Building an Equivalence

Unlike the operadic case, in general we have no reason to expect algebraic Koszul duality to be an equivalence.

Example 5. *In the case of* $Lie^! = Comm^*[1]$*, the Bar functor sends a Lie algebra to its Chevalley complex, and this functor is clearly not fully faithful: take say* sl_2*, then its Chevalley complex is concentrated on degree* (-3)*, but the trivial Lie algebra* $k[3]$ *would have the same Chevalley complex.*

Nevertheless, [FG12] proposes a conjecture about how to make this an equivalence. We say an O-algebra A is nilpotent if there exists an N such that $n > N$ implies $O(n) \otimes A^n \to A$ is zero (nulhomotopic), and we define $O\text{-alg}^{\text{nil}}(\mathcal{C})$ to be the subcategory spanned by objects that are limits of nilpotent algebras (we call such objects *pro-nilpotent*).

Observe that the coBar functor lands in this subcategory: write $O^! = \text{colim}_k O^{!,\le k}$, where $O^{!,\le k}$ is obtained by erasing $O^!(s)$ terms for all $s > k$. For $B \in O^!\text{-coalg}^{\text{nil}}_{\text{d.p.}}(\mathcal{C})$ and $A = \text{coBar}_{O^!}(B)$, define $A^{\le k} := \text{coBar}_{O^{!,\le k}}(B)$, then one can check that $A = \lim_{O\text{-alg}(\mathcal{C})}(A^{\le k})$ and $O(s)$ acts on $A^{\le k}$ by zero for $s > k$. So by adjunction, the functor $\text{Bar}^{\text{naive}}_O$ factors as $\overline{\text{Bar}^{\text{naive}}_O} \circ \text{compl}_{\mathcal{O}}$, where the completion functor $\text{compl}_{\mathcal{O}}$ is the left adjoint to the limit-preserving embedding $O\text{-alg}^{\text{nil}}(\mathcal{C}) \to O\text{-alg}(\mathcal{C})$ and $\overline{\text{Bar}^{\text{naive}}_O} : O\text{-alg}^{\text{nil}}(\mathcal{C}) \to O^!\text{-coalg}^{\text{nil}}_{\text{d.p.}}(\mathcal{C})$.

Conjecture 2 ([FG12])**.**

$$\overline{Bar^{naive}_O} : O\text{-}alg^{nil}(\mathcal{C}) \rightleftarrows O^!\text{-}coalg^{nil}_{d.p.}(\mathcal{C}) : coBar_{O^!}$$

is an equivalence of categories.

Remark 4. *0-connected case for modules over a commutative ring spectrum is proven in [CH15].*

This can be understood as a generalization of the classical results in [BGS96] of the auto-equivalence of left finite Koszul algebras.

3.3.2 Koszul Duality for E_1 Algebras

Let us look at E_1-algebras, i.e. the case of associative algebras in Vect.

Theorem 3.1 ([Lur11, Corollary 3.1.15])**.** *Let A be an E_1-algebra. If A is coconnective and locally finite, then A is Koszul.*

Note that coconnective means $\pi_0(A) = k$, $\pi_i(A) = 0$ for $i > 0$, and locally finite means $\dim \pi_i(A) < \infty$ for each i. In the classical setting this simply means our A is Artinian; in the dg setting, it means that our algebra is connective and has finite dimensional cohomologies.

Sample Computation Let's do a concrete example with chain complexes. Consider $k[x]$ for x in degree -1, so it is the complex $0 \to k \to k \to 0$ concentrated in degree 0 and -1. Let's compute what the (associative) Koszul dual $k \otimes^L_{k[x]} k$ is. The complex k (concentrated on degree 0) admits the following resolution

$$\ldots \to k[x][2] \to k[x][1] \to k[x] \to k$$

where the maps between complexes are given by

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \longrightarrow & k & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow{\scriptstyle id} & & \downarrow & & \\ & & 0 & \longrightarrow & k & \longrightarrow & k & \longrightarrow & 0 \end{array}$$

Thus we can compute the derived tensor product as

$$\text{Tot}(\ldots \to k[2] \to k[1] \to k)$$

which is given by

$$\ldots \to 0 \to k \to 0 \to k$$

i.e. $k[y]$ for y placed in degree -2. Now we compute $\text{coBar}(k[y]) = \text{Hom}_{k[y]\text{-comod}}(k,k) = \text{Hom}_{k[y^*]\text{-mod}}(k,k)$ where y^* is on degree 2. We use the following resolution:

$$0 \to k[y^*][-2] \to k[y^*] \to k \to 0$$

So the derived hom is given by

$$\mathrm{Tot}(k \to k[2] \to 0)$$

which is $k[x]$ again. More generally, if we place a vector space V on degree -1, then the trivial $\mathrm{Sym}(V[1])$-module admits the following resolution:

$$\dots \bigwedge^2 (V[1]) \otimes \mathrm{Sym}(V[1]) \to V[1] \otimes \mathrm{Sym}(V[1]) \to \mathrm{Sym}(V[1]) \to k \to 0$$

From which we can derive that $\mathrm{Sym}(V[1])^! = \mathrm{Sym}(V[2])$, considered as a coalgebra.

The Case of Lie Algebras The computation above is the abelian case of the general computation for Lie algebras. Namely, given a (dg) Lie algebra $\mathfrak{g}$, the Bar construction computes its Chevalley complex, which could be obtained from the following resolution of the trivial module:

$$\dots \bigwedge^2 (\mathfrak{g}) \otimes U(\mathfrak{g}) \to \mathfrak{g} \otimes U(\mathfrak{g}) \to U(\mathfrak{g}) \to k \to 0$$

Let us briefly explain the case of Lie algebra Koszul duality. Because there is an operad morphism Lie $\to$ Ass, we have a natural morphism $\mathrm{res} : \mathrm{Ass}(\mathcal{C}) \to \mathrm{Lie}(\mathcal{C})$, which admits a left adjoint $U : \mathrm{Lie}(\mathcal{C}) \to \mathrm{Ass}(\mathcal{C})$, and we have

$$[1] \circ \mathrm{Bar}_{\mathrm{Ass}} \circ U \simeq \mathrm{oblv}^{\mathrm{Cocomm} \to \mathrm{Coass}} \circ [1] \circ \mathrm{Bar}_{\mathrm{Lie}}$$

as functors $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Coassoc}(\mathcal{C})$. This is a lossy functor, however: whereas $U(\mathfrak{g})$ is a cocommutative Hopf algebra, now we only have a coassociative coalgebra.

To make this precise, define $\mathrm{CocommBialg}(\mathcal{C}) := E_1\text{-alg}(\mathrm{Cocomm\text{-}alg}(\mathcal{C})) \simeq \mathrm{Cocomm\text{-}alg}(E_1\text{-alg}(\mathcal{C}))$—note that this equivalence is not automatic and is checked in [GR, IV.2], and further define $\mathrm{CocommHopf}(\mathcal{C}) := \mathrm{Grp}(\mathrm{Cocomm\text{-}alg})(\mathcal{C})$. To upgrade to an equivalence of cocommutative Hopf algebras one has to loop our Lie algebra; namely we can upgrade U to $U^{\mathrm{Hopf}} : \mathrm{Lie}(\mathcal{C}) \to \mathrm{CocommHopf}(\mathcal{C})$, and we have

$$[1] \circ \mathrm{Grp}(\mathrm{Bar}_{\mathrm{Lie}}) \circ \Omega_{\mathrm{Lie}} \simeq U^{\mathrm{Hopf}}$$

as functors $\mathrm{Lie}(\mathcal{C}) \to \mathrm{CocommHopf}(\mathcal{C})$. (One might be worried that the Lie structure becomes trivial after the looping, but the Lie bracket can be recovered from the homotopy data.) The same story holds for E_n-algebras in place of E_1, the only difference being that we have to loop n times instead.

3.4 Koszul Duality for Modules

Now let $\mathcal{M}$ be the same as in section 1.3. By taking left adjoint to the trivial module functor $\mathcal{M} \to \mathrm{Mod}_A(\mathcal{M})$ we obtain another Bar functor, and similarly a cobar functor. By same reasoning as in the algebra case, this pair factors through another pair:

$$\overline{\mathrm{Bar}_A} : \mathrm{Mod}_A^{\mathrm{nil}}(\mathcal{M}) \rightleftarrows \mathrm{Comod}^{\mathrm{nil}}_{A^!,\mathrm{d.p.}}(\mathcal{M}) : \mathrm{coBar}_{A^!}$$

which we call the *modular Koszul duality.* (Warning: this is slightly different from the one in [FG12, Section 7], where they used what we write as $\mathrm{Bar}_O^{\mathrm{naive}}$. When $\mathcal{C}$ is pro-nilpotent, however, those two notions will agree.) Even if A is Koszul, there is no guarantee that its modular Koszul duality is an equivalence. However,

Proposition 5. *When $\mathcal{M}$ is pro-unipotent, these are equivalences.*

For the case of one-sided modules we have the following result:

Theorem 3.2 ([Lur11, 3.5.2])**.** *For A a small E_1-algebra (defined in [Lur11, 1.1.11]), there is an equivalence between the category of* ind-coherent *left/right modules (ind-object over small modules, i.e. those whose homotopy groups are finite dimensional) over A and that of left/right comodules over $A^!$.*

4 More on E_n Operads

The following, known as *Dunn Additivity*, is the key fact that makes things work:

Theorem 4.1 ([Dun88], [Lur, 5.1.2.2]). *For any n, m, we have E_{n+m}-alg$(\mathcal{C}) = E_n$-alg$(E_m$-alg$(\mathcal{C}))$.*

We will not try to prove this theorem, but let us mention that this has a generalization to factorization algebras. Namely it would follow from Lurie's result (locally constant factorization algebras on $\mathbb{R}^n$ are the same as E_n algebras) and the following statement:

Theorem 4.2 ([Roz]). *For any manifolds M, N, the factorization algebras on M valued in factorization algebras on N are the same as factorization algebras on $M \times N$.*

In terms of left-right modules, E_k algebra also behave well (everything below would also hold for RMod):

Corollary 1 ([Lur, 4.8.5.20]). *For A an E_n-algebra and $\mathcal{M}$ as in section 1.3, LMod$_A(\mathcal{M})$ (where A is viewed as an E_1-algebra) are E_{n-1}-categories.*

In fact something stronger is true:

Corollary 2. *If $\mathcal{M}$ is such that for every $A \in E_n$-alg$(\mathcal{C})$, there exists $M_A \in$ LMod$_A(\mathcal{M})$ such that $A \simeq$ End$_A(M_A)$, then the functor LMod$_\bullet(\mathcal{M})$ is a fully faithful functor from E_n-alg$(\mathcal{C})$ to E_{n-1}-alg$(\mathcal{C}$-ModCat).*

In particular this is satisfied by $\mathcal{M} = \mathcal{C}$ by taking $M_A = A$. In other words, specifying an E_n-algebra structure on A is equivalent to specifying an E_1-structure on A and an E_{n-1} structure on the representation category LMod$_\mathcal{C}(A)$.

Example 6. *If A is an E_3 algebra, i.e. quasi-triangular Hopf algebra, then its module category is a braided monoidal (E_2) category.*

Now if we have a E_1-algebra A, its left module category would have no monoidal structure; however, its *bimodule* category would again have an E_1 structure. The general statement is the following:

Theorem 4.3 ([Lur, 3.4.4.6]). *For $\mathcal{M} = \mathcal{C}$ and $A \in E_n$-alg$(\mathcal{C})$, we have Mod$_A(\mathcal{C}) \in E_n$-alg(A-ModCat).*

Remark 5. *The theorem is true more generally for O a coherent operad, as defined in [Lur, 3.3.1]. Also it should be straightforward to separate the exact condition on $\mathcal{M}$ for this to hold.*

4.1 (Co)Hochschild (Co)homology

Notice that when we take $\mathcal{M} = \mathcal{C}$, we have in particular $A \in \mathrm{Mod}_A(\mathcal{C})$, so it makes sense to discuss

$$HH^*(A) := \mathrm{Hom}_{\mathrm{Mod}_A(\mathcal{C})}(A, A)$$

and

$$HH_*(A) := A \otimes_{\mathrm{Mod}_A(\mathcal{C})} A.$$

We shall refer to them as the Hochschild cohomology/homology of A respectively. Dually we can define $\mathrm{CHH}^*(A)$ and $\mathrm{CHH}_*(A)$, the coHochschild cohomology/homology of a coalgebra. The following statement is usually referred to as (higher) *Deligne Conjecture*:

Proposition 6 ([Lur09, 2.5.13], [KS00], [Tam03]). *Hochschild cohomology of an E_n-algebra is an E_{n+1}-algebra.*

Example 7. *For C a monoidal category, its Hochschild cohomology would be E_2; this is the Drinfeld center.*

5 Koszul Duality for E_2 Algebras and Modules

Define Bialg($\mathcal{C}$), the category of bialgebras in $\mathcal{C}$, to be

$$E_1\text{-alg}(E_1^*\text{-coalg}(\mathcal{C})) \simeq (E_1^*\text{-coalg}(E_1\text{-alg}(\mathcal{C}))$$

(That these two definitions are equivalent is again not obvious.) Let Hopf($\mathcal{C}$) denote the full subcategory of Hopf algebra objects.

Remark 6. *Let us admit that we do not yet have a workable ∞-definition for Hopf($\mathcal{C}$), so the following can only be understood at the dg level. (In an earlier version of this note an incorrect definition was given.)*

Using additivity, we can write E_2-alg($\mathcal{C}$) as E_1-alg(E_1-alg($\mathcal{C}$)); applying Koszul duality on the inner level, we end up producing an element of Bialg($\mathcal{C}$). This observation (that the E_1 Koszul dual of an E_2-algebra is a bialgebra) was due to Tamarkin.

We give two proofs for the case $\mathcal{C}$ = Vect.

Proof by Tannakian Formalism. For any E_2-algebra A, recall that A-mod(Vect) is an E_1-algebra in DGCat, i.e. a monoidal DG category. Now apply modular Koszul to A-mod; in nice cases, this gives us $A^!$-comod for $A^! \in E_1^*[1]$-coalg, and by our remark above, the E_1 (monoidal) structure on A-mod gives a monoidal structure on $A^!$-comod. Furthermore, by definition, shift by 1 gives an isomorphism $A^!$-comod $\simeq$ ($A^![1]$)-comod, equipped with an E_1 structure. Since it also comes with a monoidal forgetful map to the underlying Vect, by general Tannakian formalism we can reconstruct the bialgebra $A^![1]$. □

Original Proof by Tamarkin. For any given operad $O \in$ Oprd(Vect), the homology of O (with trivial differential) is again an operad, which we call the *homology operad* of O and denote by HO. The key fact is the following, which is usually referred to as *Kontsevich formality*:

Theorem 5.1 ([Tam03], [Kon97])**.** *$E_n \simeq HE_n$.*

The operad HE_n is P_n, the operad of Poisson n-algebras, that is, Poisson algebras whose brackets has degree $(1-n)$. Next, there is a combinatorially defined operad B_∞, that of the brace algebras.

Proposition 7 ([KS00])**.** *$B_\infty \simeq HB_\infty \simeq P_2$.*

This means that any E_2-algebra is automatically equipped with a B_∞-algebra structure. Finally, an explicit check (e.g. [Foi17]) shows that Bar construction maps B_∞-algebras to Hopf algebras. □

Let us mention in the passing that ideas here also give another proof of the Etingof-Kazhdan quantization theorem, as noted by [Tam07]. Namely, if $\mathfrak{g}$ is a Lie bialgebra, then Sym($\mathfrak{g}[-1]$) has, by definition, the structure of an P_2-algebra; then the procedure here would yield a (dg) Hopf algebra. One then checks that the resulting Hopf algebra is concentrated on degree 0, and the degree 0 piece is a bona fide Hopf algebra, which we denote by $Q(\mathfrak{g})$. Then the Etingof-Kazhdan quantization $U_\hbar(\mathfrak{g})$, as a Hopf algebra (see below), is given as $\varprojlim_n \mathfrak{g} \otimes k[t]/t^n$.[1]

Remark 7. *The equivalence $B_\infty \simeq P_2$ implicitly involves the choice of an associator.*

Remark 8. *Under additional conditions, this procedure can in fact produce a Hopf algebra (i.e. we get the antipode map). For instance, Tannakian formalism recovers the Hopf algebra structure if the module category turns out to be rigid; likewise, if the Lie bialgebra $\mathfrak{g}$ is conilpotent (i.e. $x \mapsto \delta(x) - (1 \otimes x + x \otimes 1)$ is a nilpotent operator), then the resulting bialgebra is also conilpotent, thus equipped with an antipode structure. In particular, this is satisfied by $\mathfrak{g} \otimes k[t]/t^n$ mentioned above.*

[1] $\mathfrak{g} \otimes k[t]/t^n$ is the Lie bialgebra over $k[t]/t^n$, equipped with the same Lie bracket and the cobracket $\delta(x \otimes a) = ta\delta(x)$, $\delta(x)$ being the Lie cobracket on $\mathfrak{g}$.

6 The General Case for E_n

Finally we list some facts about general E_n algebras and modules.

Proposition 8. *Under the identification E_n-alg $\simeq E_1$-alg(E_1-alg($\ldots$)), applying the E_n Koszul duality is the same thing as applying the E_1 Koszul duality on each of the E_1-structures.*

Proposition 9 ([Lur11, 4.4.5])**.** *Let A be an E_n-algebra that is n-coconnective (meaning $\pi_i = 0$ for $i \geq n$) and locally finite. Then A is Koszul.*

Proposition 10 ([AF14])**.** *$HH_*(A) \simeq CHH_*(A^!)$ for $A \in E_n$-alg that is $(-n)$-coconnective.*

References

[AF14] David Ayala and John Francis. “Poincaré/Koszul duality”. In: (2014).

[BGS96] Alexander Beilinson, Victor Ginzburg, and Wolfgang Soergel. “Koszul Duality Patterns in Representation Theory”. In: *Journal of the American Mathematical Society* 9.2 (1996), pp. 473–527. ISSN: 0894-0347. DOI: 10.1090/s0894-0347-96-00192-0. URL: http://dx.doi.org/10.1090/s0894-0347-96-00192-0.

[CH15] Michael Ching and John E Harper. “Derived Koszul duality and TQ-homology completion of structured ring spectra”. In: (2015).

[Cos13] Kevin Costello. “Supersymmetric gauge theory and the Yangian”. In: (2013).

[Dun88] Gerald Dunn. “Tensor product of operads and iterated loop spaces”. In: *Journal of Pure and Applied Algebra* 50.3 (1988), pp. 237–258. ISSN: 0022-4049. DOI: 10.1016/0022-4049(88)90103-x. URL: http://dx.doi.org/10.1016/0022-4049(88)90103-x.

[FG12] John Francis and Dennis Gaitsgory. “Chiral Koszul duality”. In: *Sel Math* 18.1 (2012), pp. 27–87. ISSN: 1022-1824. DOI: 10.1007/s00029-011-0065-z.

[Foi17] Loic Foissy. “Algebraic structures associated to operads”. In: (2017).

[Fre11] Benoit Fresse. “Koszul duality of En-operads”. In: *Sel Math* 17.2 (2011), pp. 363–434. ISSN: 1022-1824. DOI: 10.1007/s00029-010-0047-6.

[GJ94] Ezra Getzler and John DS Jones. “Operads, homotopy algebra and iterated integrals for double loop spaces”. In: (1994).

[GK94] Victor Ginzburg and Mikhail Kapranov. “Koszul duality for operads”. In: *Duke Mathematical Journal* 76.1 (1994), pp. 203–272. ISSN: 0012-7094. DOI: 10.1215/s0012-7094-94-07608-4. URL: http://dx.doi.org/10.1215/s0012-7094-94-07608-4.

[GR] Dennis Gaitsgory and Nick Rozenblyum. *A Study in Derived Algebraic Geometry*.

[Kel99] Bernhard Keller. “Introduction to A-infinity algebras and modules”. In: (1999).

[Kon97] Maxim Kontsevich. “Deformation quantization of Poisson manifolds, I”. In: (1997). DOI: 10.1023/b:math.0000027508.00421.bf. URL: http://dx.doi.org/10.1023/b:math.0000027508.00421.bf.

[KS00] Maxim Kontsevich and Yan Soibelman. “Deformations of algebras over operads and Deligne’s conjecture”. In: (2000).

[Lef03] Kenji Lefévre-Hasegawa. “Sur les A-infini catégories”. In: (Mar. 2003).

[Lur] Jacob Lurie. *Higher Algebra*.

[Lur09] Jacob Lurie. “Derived Algebraic Geometry VI: E[k]-Algebras”. In: (Oct. 2009).

[Lur11] Jacob Lurie. “Derived Algebraic Geometry X: Formal Moduli Problems”. In: (Nov. 2011).

[Roz] Nick Rozenblyum. *Topological Chiral Categories*. URL: http://www.iecl.univ-lorraine.fr/~Sergey.Lysenko/notes_talks_winter2018/T-5(Nick).pdf.

[Tam03] Dmitry E. Tamarkin. “Formality of Chain Operad of Little Discs”. In: *Letters in Mathematical Physics* 66.1/2 (Mar. 2003), pp. 65–72. ISSN: 0377-9017. DOI: 10.1023/b:math.0000017651.12703.a1. URL: http://dx.doi.org/10.1023/b:math.0000017651.12703.a1.

[Tam07] Dmitry Tamarkin. “Quantization of Lie bialgebras via the formality of the operad of little disks”. In: *GAFA Geometric And Functional Analysis* 17.2 (2007), pp. 537–604.