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1 Chern-Simons theory

Our first goal is to formulate the flatness equation for a connection of $Ad(\mathfrak{g})$ bundle over $\mathbb{R}^3$ in terms of pertrubative field theory.

In order to do this we fix a f connection ∇_0 and consider other connections as a pertrubation as $\nabla_0 + A$ with $A \in \Omega^1(\mathbb{R}^3, \mathfrak{g})$. Now we want to write down a local action functional $S(A) = \int \mathcal{L}(A, dA)$ such that the equations of motion following form it are precisely:

$$dA + A \wedge A = 0 \ ,$$

note that $2A \wedge A = [A, A]$.

Fix a basis in $\mathfrak{g}$ – t^a. We can write down $A = A_a t^a$ From variang the action with respect to we can see that $\delta S = \int \delta A_a \wedge \frac{\delta \mathcal{L}}{\delta A_a} + \delta dA_a \wedge \frac{\delta L}{\delta (dA_a)} = \int \delta A_a \wedge (\frac{\delta \mathcal{L}}{\delta A_a} + d(\frac{\delta \mathcal{L}}{\delta dA_a}))$, since δA is arbitrary 1-form, and $\int$ gives a non-degenerate pairing $\Omega^1 \times \Omega^3 \to \mathbb{R}$ it follows that $\frac{\delta \mathcal{L}}{\delta A_a} + d(\frac{\delta \mathcal{L}}{\delta dA_a}) = 0$. Now since our equation is under the conjugation by elements of G it follows that $\mathcal{L}$ should also be invariant under it, so it should contain only invariant functions of t^a. The most natural choice for this would be taking the Killing form, which we will denote by $Tr_{\mathfrak{g}}$ since we can think about it as taking trace in a certain representation. So our ansatz for $\mathcal{L}$ becomes $\mathcal{L} = Tr_{\mathfrak{g}}(\mathcal{L}')$, thus we can rewrite $\delta S = \int Tr_{\mathfrak{g}}(\delta A \wedge (\frac{\delta \mathcal{L}'}{\delta A} + d(\frac{\delta \mathcal{L}'}{\delta dA})) = 0$, and now since $Tr_{\mathfrak{g}}$ is also non-degenerate pairing it follows that $\frac{\delta \mathcal{L}'}{\delta A} + d(\frac{\delta \mathcal{L}'}{\delta dA}) = 0$.

Now the term $A \wedge A$ can only appear from $\delta \mathcal{L}'/\delta A$, so $\mathcal{L}$' contains a term $A \wedge A \wedge A$. The term dA can appear from both places, from the first summand it can appear as $\delta(A \wedge dA)/\delta A$ and from the second as $d(\delta(dA \wedge A)/\delta dA)$. But these terms are the same, hence our $\mathcal{L}'$ is equal to:

$$cs(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A \ .$$

Also note that we can act on the connection by a local symmetry $A \mapsto gAg^{-1} + gdg^{-1}$. In the terms of Lie algebra we have $A \mapsto A + dX + [X, A]$, for $X \in \Omega^0(\mathbb{R}^3, \mathfrak{g})$. Let's see what this does with our theory. Since we already calculated $\delta S/\delta A$ lets use this:

$$\delta S = 2 \int Tr_{\mathfrak{g}}((dX + [X, A]) \wedge (dA + A \wedge A)) =$$

$$= 2 \int Tr_{\mathfrak{g}}(d(X \wedge dA) + d(X \cdot A \wedge A) + X \cdot A \wedge A \wedge A - X \cdot A \wedge A \wedge A) = 0 \ ,$$

So action functional is invariant under all such transformation.

2 4d theory on $\mathbb{C} \times \mathbb{R}^2$

Now the idea is to find a 4d perturbative field theory which would describe for us partially flat pertrubations:

$$dz \wedge F(A) = 0 \ .$$

The most obvious idea to take $\mathcal{L}' = dz \wedge cs(A)$ works and we get the following definition:

Definition 1. The four-dimensional Yangian theory on $\mathbb{C}\times\mathbb{R}^2$ is given by the following data: the space of fields is $A\in\Omega^1(\mathbb{C}\times\mathbb{R}^2)\otimes\mathfrak{g}$ such that $i_{\partial_z}A=0$ and the action is given by:

$$S(A)=\int dz\wedge CS(A)=\int dz\wedge\left\langle A,dA+\frac{1}{3}[A,A]\right\rangle_{\mathfrak{g}}=\int dz\wedge Tr_{\mathfrak{g}}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right).$$

Remark 1. We will sometimes use the notation $Tr_{\mathfrak{g}}$ instead of $<,>_{\mathfrak{g}}$ for invariant form on $\mathfrak{g}$. This makes complete sense for $\mathfrak{g}$ – reductive, since the invariant form can be given as a trace of certain representation.

Remark 2. Note that the same action can be written for $A\in\Omega^1(\mathbb{C}^2)\otimes\mathfrak{g}$ without any additional constraint, but this would lead to a theory invariant under a gauge transformation $A\mapsto A+fdz$, so we will work the theory above which is obtained by fixing the gauge to be $A_z=0$.

Now we can derive the equations of motion for this theory:

$$\delta S=\int dz\wedge Tr_{\mathfrak{g}}\left(\delta A\wedge dA+A\wedge d\delta A+2\delta A\wedge A\wedge A\right)=\int dz\wedge Tr_{\mathfrak{g}}\left(2\delta A\wedge[dA+A\wedge A]\right)=0\,,$$

since we have $\int dz\wedge Tr_{\mathfrak{g}}(A\wedge d\delta A)=-\int Tr_{\mathfrak{g}}(d(dz\wedge A)\wedge\delta A)=\int dz\wedge Tr_{\mathfrak{g}}(\delta A\wedge dA)$ .Thus we get $\int dz\wedge Tr_{\mathfrak{g}}(\delta A\wedge F(A))=0$, since the form is non-degenerate, this leads to three equations:

$$dz\wedge d\bar{z}\wedge F(A)=0\ ,dz\wedge dw\wedge F(A)=0\ ,dz\wedge d\bar{w}\wedge F(A)=0\ ,$$

which are explicitly given by:

$$\begin{aligned}\partial_w A_{\bar{w}}-\partial_{\bar{w}}A_w+[A_w,A_{\bar{w}}]=0\\ \partial_{\bar{z}}A_{\bar{w}}-\partial_{\bar{w}}A_{\bar{z}}+[A_{\bar{z}},A_{\bar{w}}]=0\\ \partial_w A_{\bar{z}}-\partial_{\bar{z}}A_w+[A_w,A_{\bar{z}}]=0\end{aligned}$$

The first equation means that A defines a flat connection over any $\{pt\}\times\mathbb{C}$. Second and third equation mean that then we go along the antiholomorphic direction in the first coordinate everything that happens is a gauge transformation of 2d connection with gauge function $A_{\bar{z}}$. So the connection varies holomorphically in the first coordinate. Dually one can think about holomorphic connection on $\mathbb{C}\times\{pt\}$ given by $d\bar{z}A_{\bar{z}}$ which varies in a flat way in the second coordinate since going in any direction produces a gauge transformation by linear combination of A_w and $A_{\bar{w}}$.

Now we would like to understand what are the gauge symmetries of $S(A)$. In order to understand this let's rewrite the action in the following form:

$$-\int zTr_{\mathfrak{g}}(F(A)\wedge F(A))=-\int zTr_{\mathfrak{g}}(dA\wedge dA+dA\wedge A\wedge A+A\wedge A\wedge dA+A\wedge A\wedge A\wedge A)=$$

$$=-\int zTr_{\mathfrak{g}}\left(dA\wedge dA+\frac{2}{3}d(A\wedge A\wedge A)\right)=S(A)\ .$$

Here we've used the fact that $A^{\wedge 4}=0$. From this form of action it follows that gauge transformations should leave $F(A)$ invariant. Hence they are given by $X\in\Omega^0(\mathbb{C}^2)\otimes\mathfrak{g}$, acting by:

$$A\mapsto A+[X,A]+dX\ .$$

Remark 3. Note that since we fixed $A_z=0$ we need to follow the above gauge transformation with another one sending $A_z\to A_z-\partial_z X$.

For the next generalization of our theory we would like to reformulate it in the following terms:

Definition 2. The *Yangian deformation of holomorphic BF theory* on $\mathbb{C}\times\mathbb{R}^2$ is the theory whose fields are $\mathcal{A}\in\Omega^{0,1}(\mathbb{C}\times\mathbb{R})\otimes\mathfrak{g}$ and $\mathcal{B}\in\Omega^{2,0}(\mathbb{C}\times\mathbb{R}^2)\otimes\mathfrak{g}$ and the action is:

$$S(\mathcal{A},\mathcal{B})=\int Tr_{\mathfrak{g}}\left(\mathcal{B}\wedge F(\mathcal{A})+\frac{\lambda}{2}dz\wedge\mathcal{A}\wedge\partial\mathcal{A}\right)\ ,$$

where λ is coupling constant. Note that $\mathcal{A}\wedge\partial\mathcal{A}=\mathcal{A}\wedge d\mathcal{A}$.

Remark 4. This theory for $\lambda=0$ turns into a holomorphic BF theory which we might discuss later. Here A describes pertrubations of holomorphic connections. These theory is a "cotangent" theory since field A and B can be thought as a dual pair in dgla $\Omega^{0,*}(\mathbb{C}^2,\mathfrak{g})$ and $\Omega^{*,0}(\mathbb{C}^2,\mathfrak{g})$ there the second one describes a shifted cotangent bundle to the moduli space of holomorphic connections. This statements will become clearer in the next talks.

We can easily show that if we write $\mathcal{B}=B_0dzdw$ and we take $B=\lambda^{-1}B_0dw$ and $A=B+\mathcal{A}$ we get the action of the original theory. Indeed:

$$\begin{aligned}S(\mathcal{A},B)=\lambda\int Tr_{\mathfrak{g}}(dz\wedge B\wedge d\mathcal{A}+dz\wedge B\wedge\mathcal{A}\wedge\mathcal{A}+1/2\mathcal{A}\wedge d\mathcal{A})=\\=\frac{\lambda}{2}\int dz\wedge Tr_{\mathfrak{g}}(B\wedge d\mathcal{A}+\mathcal{A}\wedge dB+\mathcal{A}\wedge d\mathcal{A}+2B\wedge\mathcal{A}\wedge\mathcal{A})=\\=\frac{\lambda}{2}\int dz\wedge Tr_{\mathfrak{g}}((\mathcal{A}+B)\wedge d(\mathcal{A}+B)+2/3(\mathcal{A}+B)^{\wedge 3})=\frac{\lambda}{2}S(A)\ ,\end{aligned}$$

since $dB\wedge B=B\wedge B=0$ and $A^{\wedge 3}=0$.

We can rewrite the gauge transformations as follows:

$$A\mapsto A+\bar{\partial}X+[X,A]\quad,B\mapsto B+\lambda dz\wedge\partial X\ .$$

3 Theory on complex surface

We can generalize the previous construction in the following way. Suppose X is a complex surface, D is a reduced divisor, ω is nowhere vanishing element of $K_X(2D)$, and V is a holomorphic vector field preserving D and $\mathcal{L}_V(\omega)=0$. We also fix a lift of $V\in TX$ to $At(P)=TP/G$ which we denote by ∇_V.

More precisely we consider an exact sequence of bundles: $0\to\mathfrak{g}(P)\to TP/G=At(P)\to Vect(X)$, there $Vect(X)$ is the bundle of holomorphic vector fields. The splitting of this sequence gives us a holomorphic connection, so we can think about lifting a single holomorphic vector field to be a partial connection. But we also need to consider the modified short exact sequence $\mathfrak{g}(P)(-D)\to At(P,D)\to Vect(X,D)$, there we restricted ourselves to field parallel to D. We can further restrict ourselves to $\mathfrak{g}(P)(-D)\to At(P,D)^{div}\to Vect^{div}(X,D)$, there we restrict ourselves to fields which fix ω. So ∇_V is a lift of V in the latter sequence.

Definition 3. The Yangian deformation of holomorphic BF theory is a theory where fields are $\alpha\in\Omega^{0,1}(X,\mathfrak{g}_P(-D))$ and $\beta\in\Omega^{0,0}(X,\mathfrak{g}_P(-D))$, where $\mathfrak{g}_P$ is a Lie algebra bundle associated with P and $\mathfrak{g}_P(-D)=\mathfrak{g}_P\otimes O(-D)$. The action is given by:

$$S(\alpha,\beta)=\int_X\omega\wedge Tr_{\mathfrak{g}}\left(-\alpha\wedge\bar{\partial}\beta+\frac{\lambda}{2}\alpha\wedge\nabla_V\alpha+\beta\cdot\alpha\wedge\alpha\right)\ .$$

First, it is easy to see that if $X = \mathbb{C}^2$, $D = 0$, $\omega = dz \wedge dw$, $V = \partial_w$ and the bundle is trivial with trivial connection we get our previous theory under the identification $\alpha = \mathcal{A}$ and $\omega\beta = \mathcal{B}$. The first and the thrid terms give us:

$$\int_X \omega \wedge Tr_{\mathfrak{g}} (-\alpha \wedge d\beta + \beta \cdot \alpha \wedge \alpha) = \int Tr_{\mathfrak{g}}(\mathcal{B} \wedge [d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}]) \ .$$

And the middle term is:

$$\int_X \omega \wedge Tr_{\mathfrak{g}}(\alpha \wedge i_V d\alpha) = \int i_V(\omega) \wedge Tr_{\mathfrak{g}}(\alpha \wedge d\alpha) = \int dz Tr_{\mathfrak{g}}(\mathcal{A} \wedge \partial\mathcal{A}) \ ,$$

so the sum is indeed $S(\mathcal{A}, \mathcal{B})$.

Now we can derive the equations of motions. First using $\int \omega Tr_{\mathfrak{g}}(\alpha \wedge \partial\beta) = \int \omega Tr_{\mathfrak{g}}(\beta \wedge d\alpha)$ we have from variation with respect to β:

$$F^{0,2}(\alpha) = 0 \ ,$$

next using:

$$\int \omega Tr_{\mathfrak{g}}(\alpha \wedge \nabla_V \delta\alpha) = \int \omega \wedge (\alpha_a \wedge \delta\alpha_b Tr_{\mathfrak{g}}(t^a \nabla_V(t^b)) + \alpha_a \wedge \mathcal{L}_V(\alpha_b) Tr_{\mathfrak{g}}(t^a t^b)) =$$

$$= \int \omega \wedge (\delta\alpha_b \wedge \alpha_a Tr_{\mathfrak{g}}(t^b \nabla_V(t^a)) + \alpha_b \wedge \mathcal{L}_V(\alpha_a) Tr_{\mathfrak{g}}(t^b t^a)) = \int \omega \wedge Tr_{\mathfrak{g}}(\delta\alpha \wedge \nabla_V(\alpha))$$

we have:

$$\delta S = \int \omega \wedge Tr_{\mathfrak{g}}(\delta\alpha \wedge [-\bar{\partial}\beta + \lambda\nabla_V(\alpha) + [\alpha, \beta]]) = 0 \ ,$$

so $\lambda\nabla_V(\alpha) - \bar{\partial}\beta + [\alpha, \beta] = 0$.

We know what in a special case of this theory the gauge transformations are given by $\Omega^{0,0}(X, \mathfrak{g}_P(-D))$, so the good idea is to try acting on α as on usual connection form and to extend this action to some action on β in a such a way that the action is preserved.

So suppose $\delta\alpha = \bar{\partial}c + [c, \alpha]$, then:

$$\delta S = \int \omega \wedge Tr_{\mathfrak{g}}(\delta\beta \wedge F(\alpha) + (\bar{\partial}c + [c, \alpha]) \wedge (-\bar{\partial}\beta + \lambda\nabla_V(\alpha) + [\alpha, \beta])) \ .$$

Let's work on the second summand term by term, we have:

$$\int \omega \wedge Tr_{\mathfrak{g}}(dc \wedge d\beta) = \int d(\omega \wedge Tr_{\mathfrak{g}}(c d\beta)) = 0 \ ,$$

$$\int \omega \wedge Tr_{\mathfrak{g}}(dc \wedge [\alpha, \beta] - [c, \alpha] \wedge d\beta) = \int \omega \wedge (dc_i \wedge \alpha_j \beta_k - c_i \alpha_j \wedge d\beta_k) Tr_{\mathfrak{g}}(t_i [t_j, t_k]) = -\int \omega \wedge Tr_{\mathfrak{g}}(d\alpha [c, \beta])$$

$$\int \omega \wedge Tr_{\mathfrak{g}}([c, \alpha] \wedge [\alpha, \beta]) = -\int \omega \wedge Tr_{\mathfrak{g}}(\alpha \wedge [[c, \alpha], \beta] + \alpha \wedge [\alpha, [c, \beta]]) = \int \omega \wedge Tr_{\mathfrak{g}}([\alpha, \beta] \wedge [c, \alpha] + [\alpha, \alpha] \wedge [c, \beta])$$

so:

$$\int \omega \wedge Tr_{\mathfrak{g}}([c, \alpha] \wedge [\alpha, \beta]) = -\frac{1}{2} \int \omega \wedge Tr_{\mathfrak{g}}([c, \beta] \wedge [\alpha, \alpha]) \ .$$

The next term:

$$\int \omega \wedge Tr_{\mathfrak{g}}(\bar{\partial}c \wedge \nabla_V \alpha) = \int \omega \wedge Tr_{\mathfrak{g}}(c \nabla_V \bar{\partial}\alpha) = -\int \omega \wedge Tr_{\mathfrak{g}}(\nabla_V c \wedge d\alpha) \ .$$

$$\int \omega \wedge Tr_{\mathfrak{g}}([c, \alpha] \wedge \nabla_V \alpha) = \int \omega \wedge Tr_{\mathfrak{g}}([\nabla_V c, \alpha] \wedge \alpha + [c, \nabla_V \alpha] \wedge \alpha) = \int \omega \wedge Tr_{\mathfrak{g}}([c, \alpha] \wedge \nabla_V \alpha - \nabla_V c [\alpha, \alpha]) \ .$$

So we get:

$$\delta S = \delta S = \int \omega \wedge Tr_{\mathfrak{g}}((\delta\beta - \lambda\nabla_V c - [c,\beta]) \wedge F(\alpha)) = 0 \ ,$$

hence the formula for gauge transformation of β is:

$$\beta \mapsto \beta + \lambda\nabla_V c + [c,\beta] \ .$$

Suppose X is a direct product.

So we can take $X = \Sigma \times C$, there C and Σ are complex curve. Then we can consider a divisor D is just a divisor on C times Σ, ω is ω' on C times dw and $V = \partial_w$. This theory can be then reformulated in the spirit of original Yangian theory as $\int \omega' \wedge cs(A)$, and we can suppose that Σ is an arbitrary 2-dimensional manifold. In this case the theory is connected with the following classes of quantum groups:

$$\begin{array}{llllllll} C = \mathbb{C} & , & \omega = dz & , & \text{double pole at } \infty & , & \text{(rational)} \\ C = \mathbb{C}^\times & , & \omega = \dfrac{dz}{z} & , & \text{poles at } \infty \text{ and } 0 & , & \text{(trigonometric)} \\ C = \mathbb{E} & , & \omega = dz & , & \text{no poles} & , & \text{(elliptic)} \end{array}$$

3.1 Theory on $E \times S^1 \times \mathbb{R}$

We can restrict ourselves to $E \times \mathbb{C}$, with a divisor D given by $D' \times \mathbb{C}$, $\omega = \omega' \wedge dw$, where $\omega' \in K(2D')$ and vector field by ∂_w. In this case we can also rewrite the theory in the form $\int \omega' \wedge CS(A)$, with $A \in \Omega^1(E \times \mathbb{C}, \mathfrak{g})$. So we can compactify this theory on $E \times S^1 \times \mathbb{R}$, if we now restrict ourselves to the connections constant in $\mathbb{R}$ direction we get the following equations of motion:

$$\partial_{\bar{z}} A_x - \partial_x A_{\bar{z}} + [A_{\bar{z}}, A_x] = 0 \ ,$$

this equations as before describe a flat family of holomorphic vector bundles on E. But now the can have a non-trivial holonomy, indeed by exponentiation the equation along x we get an isomorphism of holomorphic vector bundles. So we can think about solutions as pairs A, ϕ, there A is an $(0,1)$ form and ϕ is a bundle automorphism which preserves it.

Once you account for gauge transformations, the moduli space of pairs (A, φ) modulo gauge is known as the moduli space of "multiplicative Higgs" bundles.

Notice that the space of classical solutions admit a symplectic form:

$$\omega(A, A') = \int \omega \wedge Tr_{\mathfrak{g}}(A \wedge A') \ .$$

4 $N = 1$ SUSY theory

Here the sketch the way to derive the Yangian theory from the N=1 supersymmetric theory. Roughly the procedure consist of the following steps: writing down the action for the thorny, rewriting it in holomorphic terms, twisting it to get rid of supersymmetry (here the obtain the BF theory) and then deforming it (alternatively we can deform it before twisting).

So we start with a theory with a following action functional:

$$\int Tr_{\mathfrak{g}}(F(A)_+ \wedge B + cB \wedge B + \psi_+ \partial\!\!\!/_A \psi_-) \ ,$$

with $A \in \Omega^1(\mathbb{R}^4, \mathfrak{g})$, $B \in \Omega^2(\mathbb{R}^4, \mathfrak{g})_+$ – the space of self-dual forms, $\Psi_\pm \in \Omega^0(\mathbb{R}^4, \mathfrak{g} \otimes S_\pm)$ with $S_\pm$ being the images of the projection by $\frac{1\pm\omega}{2}$ ($\omega = \gamma^1\gamma^2\gamma^3\gamma^4$ – in the Clifford algebra) and finally $\not{\partial}_A = \sigma^\mu(\partial_\mu + A_\mu)$.

The equation of motion of this theory ensure that B is proportional to self dual part of $F(A)$, that both spinors satisfy the Weyl equation, and the following equation:

$$-dB + [A, B] + *(\psi_+ \sigma_\mu dx^\mu \psi_-) = 0 \ .$$

It can be easily seen that the action is invariant under the gauge transformation by $X \in \Omega^0(\mathbb{R}^4, \mathfrak{g})$ by $A \mapsto A + dX + [X, A]$, $B \mapsto B + [X, B]$, $\psi_\pm \mapsto \psi_\pm + [X, \psi_\pm]$, indeed this action comes from the group action which acts on all $F(A), B, \psi_+$ and $\not{\partial}_A \psi_-$ by conjugation, and so leaves the action invariant.

This theory can be rewritten in terms of holomorphic geometry in the following way. We choose a complex structure such that $S_- \otimes \Omega^0 = \Omega^{1,0}$, then $\Omega^2_+ = \Omega^{2,0} \oplus \omega \cdot \Omega^0 \oplus \Omega^{0,2}$ and $S_+ \otimes \Omega^0 = \Omega^0 \cdot \omega \oplus \Omega^{2,0}$. In this terms dirac operator $\not{\partial}_A : S_- \otimes \Omega^0 \to (S_+)' \otimes \Omega^0$ becomes:

$$\not{\partial}_A : \Omega^{1,0} \otimes \mathfrak{g} \xrightarrow{d_{dR} + [A, -]} \Omega^2 \otimes \mathfrak{g} \xrightarrow{\pi} \Omega^{2,0} \otimes \mathfrak{g} \oplus \omega \cdot \Omega^0 \otimes \mathfrak{g} \ .$$

And the action is now equal to:

$$\int Tr_{\mathfrak{g}}(B \wedge F(A) + \psi_+ \wedge d_A \psi_- + cB \wedge B) \ .$$

In this terms it is easy to introduce the supercharge Q acting as follows: it maps $\mathcal{S}_-$ into Ω^1 and Ω^2_+ into $\mathcal{S}_+$ by the natural maps.

Now if we do something called "twisting" to this theory we will gett BF theory, which we should later deform to get Yangian theory. Alternatively we can deform theory befory twisting, deform the charge and get Yangian theory after twisting without further deformation.

5 Physical introduction to ghosts and BRST

Let's consider a 4D Pure Yang Mills theory which is specifed by the action:

$$S(A) = -\int Tr_{\mathfrak{g}}(F(A) \wedge *F(A)) \ ,$$

the equation of motion following from this action is $d * F(A) + [A, *F(A)] = 0$. This action is obviously invariant under transformation $A \mapsto A + dX + [X, A]$. So $\mathfrak{g} \otimes \Omega^0$ is a gauge algebra.

Now one of the main physical instruments in QFT is a functional integral, which is roughly:

$$\int D\mathcal{A} \exp{[iS(A)]} f(A) \ ,$$

where A is some local gauge-invariant functional polynomial in A and it's derivatives.

The problem with this integral is that it is constant along gauge directions, so we get an infinity we don't know how to control. In order to make more sense out of it we would like to somehow integrate over the transversal to the gauge orbits and take out from the integral "the volume" of gauge group.

More concretely we would like to fix some gauge $G(A) = 0$, and use the following identity:

$$\int D\mathcal{X} \delta(G(A^X)) \det(\frac{\delta G(A^X)}{\delta X}) = 1 \ ,$$

here $A^X = A + dX + [X, A]$. This identity "holds" as an generalization of finite-dimensional identity $(\prod \int dx_i) \prod \delta(g_i(x)) \det(\partial g_i/\partial x_j)$.

If we now insert this identity in the functional integral:

$$\int D\mathcal{A}\ D\mathcal{X}\ \ \exp[iS(A^X)] f(A^X) \delta(G(A^X)) \det(\frac{\delta G(A^X)}{\delta X}) =$$

$$= (\int D\mathcal{X}) \int D\mathcal{A} \delta(G(A)) f(A) \exp[iS(A)] \det(\frac{\delta G(A^X)}{\delta X}) \ ,$$

since we are interested in ratios of functional integrals we may throw out infinite integral $\int D\mathcal{X}$.

The next trick is to notice that this works for every $G(A) - \omega(x)$. So up to infinite constant our integral is:

$$\int D\omega exp[i \int d^4x \frac{\omega^2}{2\xi}] \int D\mathcal{A} \delta(G(A) - \omega) f(A) \exp[iS(A)] \det(\frac{\delta G(A^X)}{\delta X}) =$$

$$= \int D\mathcal{A} f(A) \exp[iS(A) + i \int d^4x \frac{G(A)}{2\xi}] \det(\frac{\delta G(A^X)}{\delta X}) \ .$$

We can rewrite it further introducting a new field:

$$\int D\mathcal{A}\ D\mathcal{B}\ f(A) \exp[iS(A) - i \int d^4x \frac{B^2 - 2BG(A)}{2\xi}] \det(\frac{\delta G(A^X)}{\delta X}) \ .$$

Now the ghosts appear then we want to rewrite the determinant as a functional integral of some other theory. The idea behind the next step is that the Gaussian integral over anti-commuting variables $(\prod \int d\xi_i) \exp((\xi, \xi)_A)$ is proportional to the determinant of A.

Here we need to make a choice for G. The most natural one is $G = div(A) = \partial_\mu A^\mu$. Then the operator of which we want to calculate determinant is $\delta G(A^X)/\delta X$ can be computed as follows:

$$G(A^X) = \partial_\mu A^\mu + \partial_\mu \partial^\mu X + \partial_\mu [X, A_\mu] \ ,$$

if we denote $\partial_\mu - [A_\mu, -] = D_\mu$, then $\delta G(A^X)/\delta X = \partial_\mu D^\mu$. Let's denote this operator by D_A, then we can write:

$$\int D\bar{c} Dc \exp[\int d^4x - \bar{c} D_A c] = \det(\delta G(A^X)/\delta X) \ .$$

Thus as a result of our calculation we found that instead of working with the original action we can take the action:

$$S(A, B, \bar{c}, c) = \int Tr_{\mathfrak{g}}(F(A) \wedge F(A) + d^4x\{-\frac{B^2}{2\xi} + \frac{Bdiv(A)}{\xi} - \bar{c} Dc\} \ .$$

It turns out that there is a new symmetry acting on this theory, which is called BRST-symmetry and which sends:

$$\delta A_\mu = dc - [A, c] \ , \delta c = -\frac{1}{2}[c, c] \ , \delta \bar{c} = B, \ \ \delta B = 0 \ .$$

This symmetry also turns out to be nilpotent.