Classical Field Theory in Batalin-Vilkovisky Formalism

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In these notes we develop the Batalin-Vilkovisky formalism for the (perturbative) classical field theory.

"Perturbative" means that we will only consider those solutions which are infinitesimally close to a given solution, i.e. the formal completion at this point.

"Classical" means we are interested in the solutions to the Euler-Lagrange equations for an action functional. (We will consider the quantum effects in the next time.)

1 Formal Moduli Problem

The naive moduli space of solutions does NOT work, we need to consider the derived version. This gives a formal moduli problem.

Denote the category of Artinian dg algebra over k by $\mathbf{Art}_k$, it is simplicially enriched in a natural way.

Definition 1. *A (pointed) formal moduli problem over k is a functor (of simplicially enriched categories)*

$$F : \mathbf{Art}_k \longrightarrow sSets$$

from $\mathbf{Art}_k$ to the category of simplicial sets, with the following additional properties:

1. *$F(k)$ is contractible.*
2. *F takes surjections to Kan fibrations.*
3. *Suppose A, B, C are dg Artinian algebras with maps $B \longrightarrow A$, $C \longrightarrow A$, one of them is surjective, then we can form the fiber product $B \underset{A}{\times} C$. We require the natural map*

$$F(B \underset{A}{\times} C) \longrightarrow F(B) \underset{F(A)}{\times} F(C)$$

is a weak homotopy equivalence.

The category of formal moduli problems is naturally simplicially enriched.

One important way in which formal moduli problem arise is as the the solutions to the Maurer-Cartan equations in an L_∞ algebra.

Given an L_∞ algebra $\mathfrak{g}$ and a dg Artinian algebra $(R, \mathfrak{m})$, let

$$MC(\mathfrak{g} \otimes \mathfrak{m})$$

denote the simplicial set of solutions to the Maurer-Cartan equation in $\mathfrak{g} \otimes \mathfrak{m}$, i.e. an n-simplex is an element

$$\alpha \in \mathfrak{g} \otimes \mathfrak{m} \otimes \Omega^*(\Delta^n)$$

of cohomological degree 1 satisfying the Maurer-Cartan equation

$$\sigma_{n \geq 1} \frac{1}{n!} l_n(\alpha, \cdots, \alpha) = 0$$

Sending R to $MC(\mathfrak{g} \otimes \mathfrak{m})$ defines a formal moduli problem which we denote by $B\mathfrak{g}$.

Theorem 1. *B defines an equivalence between the $(\infty, 1)$ categories of L_∞ algebras and formal moduli problems.*

In finite dimension, $B\mathfrak{g}(R) \cong Map(C^*(\mathfrak{g}), R)$, hence we may formally think of $B\mathfrak{g}$ as $Spec\, C^*(\mathfrak{g})$. Then $\mathfrak{g}$ modules corresponds to vector bundles over $B\mathfrak{g}$, with $\mathfrak{g}[1]$ corresponding to the tangent bundle.

Definition 2. *Let M be a smooth manifold, a local L_∞ algebra on M consists of the following data:*

1. *A graded vector bundle L on M, whose space of smooth sections denoted by $\mathcal{L}$*
2. *A collection of poly-differential operators*

$$l_n : \mathcal{L}^{\otimes n} \longrightarrow \mathcal{L}$$

of degree $2 - n$ for $n \geq 1$, endowing $\mathcal{L}$ with the structure of L_∞ algebra.

Applying the functor B to $\mathcal{L}$ we get a presheaf of formal moduli problems. Because $\mathcal{L}$ is a fine sheaf, $B\mathcal{L}$ is a homotopy sheaf.

Definition 3. *An elliptic L_∞ algebra is a local L_∞ algebra as above with the property that $(\mathcal{L}, d = l_1)$ is an elliptic complex.*

Example 1. *ϕ^4 theory.*

Given a compect Riemannian manifold (M, g) the ϕ^4 theory is a scalar field theory defined by the action functional

$$S(\phi) = \int_M \frac{1}{2!}(|\nabla \phi|^2 - m^2\phi^2) - \frac{\lambda}{4!}\phi^4 \, dVol_g$$

where ϕ is a smooth function on M.

The Euler-Lagrange equation for this action functional is

$$(-\Delta + m^2)\phi + \frac{1}{3!}\phi^3 = 0$$

The actual space of solutions to this nonlinear PDE is complicated, we will describe the formal moduli problem of solutions to this equation where ϕ is infinitesimally close to 0.

This formal moduli problem of solutions to this equation can be described as the solutions to the Maurer-Cartan rquation in a certain L_∞ algebra which continue we call lc. As a cochain complex, $\mathcal{L}$ is

$$C^\infty(M)[-1] \xrightarrow{-\Delta+m^2} C^\infty(M)[-2]$$

All higher brackets are 0 except for l_3:

$$l_3 : C^\infty(M)[-1]^{\otimes 3} \longrightarrow C^\infty(M)[-2]$$
$$\phi_1 \otimes \phi_2 \otimes \phi_3 \mapsto -\lambda\phi_1\phi_2\phi_3$$

2 Classical BV Formalism

Finite dimensional toy model:

M a smooth manifold, $S \in C^\infty(M)$ an action functional, then the critical locus $Crit(S)$ is the intersection of two sections 0 and dS in T^*M.

In derived world we should take the derived intersection:

$$\begin{aligned}
\mathcal{O}(Crit(S)) &= \mathcal{O}(\Gamma(dS)) \overset{\mathbb{L}}{\underset{\mathcal{O}(T^*M)}{\otimes}} \mathcal{O}(M) \\
&= (\cdots \xrightarrow{\vee dS} \Gamma(M, \wedge^2 TM) \xrightarrow{\vee dS} \Gamma(M, TM) \xrightarrow{\vee dS} \Gamma(M, \mathcal{O}(M))) \\
&\quad (Koszul\ complex) \\
&= (\mathcal{O}(T^*[-1]M), \{S, -\})
\end{aligned}$$

where in the last line $T^*[-1]M$ denotes the shifted cotangent bundle with the natural symplectic structure of degree -1.

We can also consider the $G-$equivarient version of this construction (for simplicity assume $M = E$ is linear): in this case we are interested in the quotient $Crit(S)/G$, which is (at least on the infinitesimal level) encoded by $C^*(\mathfrak{g}, \mathcal{O}(E))$. We can view this as the functions on a supermanifold denoted by $E \oplus \mathfrak{g}[1]$. Then we take the derived intersection as above, get a super manifold $E \oplus \mathfrak{g}[1] \oplus E^*[-1] \oplus \mathfrak{g}^*[-2]$ with an odd vector field.

The four summands are called fields, ghosts, anti-fields and anti-ghosts respectively.

Motivated by the toy model above, we are now ready to present the general definition of a classical field theory:

Definition 4. *Suppose E is an elliptic L_∞ algebra on M, an invariant pairing of degree k is a symmetric map:*

$$\langle -, - \rangle_E : E \otimes E \longrightarrow Den(M)[k]$$

such that

1. *This pairing is nondegenerate, i.e. the induced map*

$$E \longrightarrow E^* \otimes Den(M)[k]$$

   is an isomorphism.

2. *$\int_M \langle \alpha, \beta \rangle_E$ is an invariant pairing on $\mathcal{E}_c$, i.e.*

   $\langle l_n(\alpha_1, \cdots, \alpha_n), \alpha_{n+1} \rangle_E$ is symmetric.

Definition 5. *A formal pointed elliptic moduli problem with a symplectic form of cohomological degree k on a manifold M is an elliptic L_∞ algebra on M with an invariant pairing of degree $k-2$.*

Definition 6. *A (perturbative) classical field theory on M is a formal pointed elliptic moduli problem on M with a symplectic form of cohomological degree -1.*

Given a local L_∞ algebra $\mathcal{L}$ over M and an action functional $S \in \mathcal{O}_{loc}(B\mathcal{L})$ (local functionals), we carry out the same procedure as in the toy model:

given an L_∞ algebra $\mathfrak{g}$, we have a universal derivation:

$$d : C^*_{red}(\mathfrak{g}) \longrightarrow C^*(\mathfrak{g}, \mathfrak{g}^*[-1])$$

, which restricts to local functionals:

$$\mathcal{O}_{loc}(B\mathcal{L}) \longrightarrow C^*_{loc}(\mathcal{L}, \mathcal{L}^![-1])$$

The derived critical locus $Crit(S)$ is $(\mathcal{L} \rtimes \mathcal{L}^![-3])$ twisted by dS. If the differential is elliptic, this defines a classical field theory.

More concisely, a classical field theory on M consists of the following data:

1. a graded vector bundle E over M with a symplectic pairing with degree -1,
2. an action functional $S \in \mathcal{O}^{\geq 2}_{loc}(\mathcal{E}(M))$ of cohomological degree 0, satisfying:
   - $\{S, S\} = 0$,
   - the quadratic part of S is elliptic.

3 Cotangent Field Theories

Definition 7. *Let $\mathcal{L}$ be an elliptic L_∞ algebra on a manifold X, and let $B\mathcal{L} = \mathcal{M}$ be the associated elliptic moduli problem. Then the cotangent field theory associated to $\mathcal{M}$is the -1-symplectic elliptic moduli problem $T^*[-1]\mathcal{M}$, whose elliptic L_∞ algebra is $\mathcal{L} \oplus \mathcal{L}^![-3]$.*

Example 2. *Anti-selfdual Yang-Mills theory:*

Let X be a oriented 4-manifold with a conformal class of a metric, G a compact Lie group and $\mathcal{M}(X, G)$ be the elliptic moduli problem parameterizing principal G-bundles on X with a connection whose curvature is anti-selfdual. Its completion near a point $(P, \nabla) \longrightarrow X$ corresponds to the L_∞ algebra

$$\Omega^0(X, \mathfrak{g}_P) \xrightarrow{d_\nabla} \Omega^1(X, \mathfrak{g}_P) \xrightarrow{d_-} \Omega^2_+(X, \mathfrak{g}_P)$$

Thus, the elliptic L_∞ algebra describing $T^*[-1]\mathcal{M}$ is given by

$$\Omega^0(X, \mathfrak{g}_P) \xrightarrow{d_\nabla} \Omega^1(X, \mathfrak{g}_P) \oplus \Omega^2_+(X, \mathfrak{g}_P) \tag{1}$$

$$\xrightarrow{d_+ \oplus d_\nabla} \Omega^2_+(X, \mathfrak{g}_P) \oplus \Omega^3(X, \mathfrak{g}_P) \xrightarrow{d_\nabla} \Omega^4(X, \mathfrak{g}_P) \tag{2}$$

The usual Yang-Mills theory is a deformation of this anti-selfdual theory by $id : \Omega^2_+(X, \mathfrak{g}_P)[-1] \longrightarrow \Omega^2_+(X, \mathfrak{g}_P)[-2]$.

4 Observables of a Classical Field Theory

Definition 8. *The observables with support in the open set U is the commutative dg algebra*

$$Obs^{cl}(U) = C^*(\mathcal{L}(U))$$

This defines a factorization algebra on M.

We want to equip it with a shifted Poisson structure, but this is somewhat subtle due to the complications that arisr when working with infinite-dimensional vector spaces. We will exhibit a sub-factorization algebra $\widetilde{Obs}^{cl}$ which is equipped with a commutative product and Poisson bracket and such that the inclusion is a quasi-isomorphism.

Observation: for the free theory we can take the subalgebra generated by smooth sections in the distributional section. By Atiyah-Bott lemma, the inclusion is a quasi-isomorphism.

However, for interaction theory this subalgebra is not preserved by higher brackets.

Instead, we need the subcomplex of functionals that have smooth first derivatives. This forms a subalgebra and is preserved by the differential. The nontrivial part is to construct the Poisson bracket. We can use the symplectic form to identify tangent bundle and cotangent bundle and pull back the Lie bracket of vector fields.(c.f. [2][1])

5 Supersymmetric Gauge Theory

In this section we consider the supersymmetric gauge theory whose quantization will lead to Yangnian.

Recall that $Spin(4) \cong SU(2) \times SU(2)$, let $S_\pm$ be the defining complex representations of the two copies of $SU(2)$, thus complex 2 dimensional, equipped with representation of $Spin(4)$. Let $V \cong \mathbb{R}^4$ be the fundamental representation of $Spin(4)$, there is an isomorphism of complex $Spin(4)$ representations:

$$\Gamma : S_+ \otimes S_- \cong V_{\mathbb{C}} \cong V \otimes \mathbb{C}$$

Definition 9. *The $N = 1$ supertranslation Lie algebra T is the complex super Lie algebra*

$$T = V_{\mathbb{C}} \oplus \Pi(S_+ \oplus S_-)$$

with bracket defined as follows: if $Q_\pm \in S_\pm$, then

$$[Q_+, Q_-] = \Gamma(Q_+ \otimes Q_-)$$

and all other brackets are zero.

T has an evident action of $Spin(4)$ as well as an action of $\mathbb{C}^*$ where the weights of $V_{\mathbb{C}}, S_+, S_-$ are $0, 1, -1$ respectively. $\mathbb{C}^*$ is called the R-symmetry group.

Definition 10. *The $N = 1$ supersymmetric gauge theory in the first order formalism has space of fields*

$$\Omega^1 \otimes \mathfrak{g} \oplus \Omega^2_+ \otimes \mathfrak{g} \oplus \Pi(\mathcal{S}_+ \oplus \mathcal{S}_-) \otimes \mathfrak{g}$$

Denote the fields in those four summands by A, B, Ψ_+, Ψ_-, the action functional is:

$$S(A, B, \Psi_+, \Psi_-) = \int \langle F(A)_+, B\rangle_{\mathfrak{g}} + c\int \langle B, B\rangle_{\mathfrak{g}} + \int \langle \Psi_+, \partial\!\!\!/_A \Psi_-\rangle_{\mathfrak{g}}$$

Here c is the coupling constant, and we are using the canonical symplectic pairing

$$\mathcal{S}_+ \otimes \mathcal{S}_+ \longrightarrow C^\infty(\mathbb{R}^4)$$

to define the action functional on the spinor.
The gauge Lie algebra is $\Omega^0 \otimes \mathfrak{g}$, the infinitesimal action of $X \in \Omega^0 \otimes \mathfrak{g}$ is given by

$$(A, B, \Psi_+, \Psi_-) \mapsto (dX + [X, A], [X, B], [X, \Psi_+], [X, \Psi_-])$$

For supersymmetry we need to define an action of $\mathcal{S}_+$ on the space of fields. This is given by

$$Q \otimes (A, B, \Psi_+, \Psi_-) \mapsto (\Gamma(Q \otimes \Psi_-), 0, Y(Q \otimes B, 0)$$

where Γ and Y are the natural maps.

Lemma 1. *This action commutes with the action of gauge Lie algebra and preserves the action functional on the space of fields.*

We can write this theory in BV formalism. This is a super version of the Yang-Mills theory.

Twisting: consider the theory defined by

$$\mathcal{L}^Q = (\mathcal{L}((t)), d + tQ)^{\mathbb{C}_R}$$

where t has cohomological degree 1, odd super degree and weight -1 under the R symmetry group.

This is a holomorphic twist (i.e. this theory has a holomorphic reformulation BF theory.)

Definition 11. *BF theory on $\mathbb{C}^2$:*

Fields: $A \in \Omega^{0,1} \otimes \mathfrak{g}$, $B \in \Omega^{2,0} \otimes \mathfrak{g}$

Action: $S = \int \langle F(A), B\rangle_{\mathfrak{g}}$

Gauge: $\Omega^{0,0} \otimes \mathfrak{g}$*, with action*

$$(A, B) \mapsto (\bar{\partial}_A X, [X, B])$$

In BV theory, this is described by

$$\mathcal{L} = \Omega^{0,*} \otimes \mathfrak{g} \oplus \Omega^{2,*} \otimes \mathfrak{g}[-1]$$

This is the cotangent theory to the derived moduli space of G-bundles on $\mathbb{C}^2$

Deformation of the theory:

$$S' = S + \frac{\lambda}{2} \int dz \langle A, \partial A\rangle_{\mathfrak{g}}$$

$$X(A, B) = (\bar{\partial}_A X, \lambda dz \wedge \partial X + [X, B])$$

In BV theory, this is described by

$$\mathcal{L} = \Omega^{0,*} \otimes \mathfrak{g} \xrightarrow{dz\partial} \Omega^{2,*} \otimes \mathfrak{g}[-1]$$

Explicit computation shows that locally the observables take the form

$$Obs^{cl} \cong C^*(\mathfrak{g}[[z]])$$

References

[1] K. Costello. *Renormalization and effective field theory*, volume 170 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2011.

[2] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory. Vol. 1,2*, volume 31 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2017.