FILTERED KOSZUL DUALITY

DMYTRO MATVIEIEVSKYI

Abstract. In this talk we define the filtered Koszul dual $A^!$ of a complete filtered augmented dg-algebra A. We will construct a quasi-equivalences between the categories A-mod and $A^!$-comod of correctly defined modules and comodules correspondingly. For an E_2-algebra A with some additional assumptions we show that $H^0(A^!)$ is a Hopf algebra. The main reference for this talk is [Cos13, Chapters 7, 8].

1. Filtered Koszul duality

1.1. **Classic Koszul duality.** Let A be an augmented E_1-algebra. We define the Koszul dual coalgebra $A^!$ as the derived functor $\mathbb{C} \otimes^{\mathbb{L}}_A \mathbb{C}$. That is a coaugmented dg-coalgebra that can be computed by the bar resolution. Please, note that in the standard definition, the Koszul dual of A should be an algebra. This algebra is linearly dual to the coalgebra $A^!$.

The following is an important theorem of Tamarkin. The goal of this talk is to prove a slightly generalized version of this theorem.

Theorem 1.1. *Let A be an E_2-algebra which is augmented as an E_1-algebra. Then $A^!$ has a natural structure of a Hopf algebra (up to coherent homotopy).*

The following example is especially interesting for us. Let $\mathfrak{g}$ be a nilpotent Lie algebra concentrated in cohomological degree 0, and $U(\mathfrak{g})$ be its universal enveloping algebra. We denote the Chevalley-Eilenberg complex of $\mathfrak{g}$ by $CE(\mathfrak{g})$. The following lemma is well-known and will be proved later in a slightly bigger generality.

Lemma 1.2. *We have an isomorphism of topological Hopf algebras $(CE(\mathfrak{g}))^! \simeq U(\mathfrak{g})^\vee$, where the latter one is the linear dual to $U(\mathfrak{g})$ with the dual Hopf algebra structure.*

Remark 1.3. *In this talk we always assume an existence of topology on all vector spaces we have. For example, we can talk about dual Hopf algebra structures without any additional assumptions.*

1.2. **Filtered Koszul duality.** We want to establish Lemma 1.2 for a general Lie algebra $\mathfrak{g}$. For this reason we will use the following approach.

Definition 1.4. *A complete filtered cochain complex is a cochain complex V with a decreasing filtration $V = F^0V \supset F^1V \supset \dots$ by sub-cochain complexes such that $V = \varprojlim V/F^iV$. Morphisms between complete filtered cochain complexes are the maps of complexes that preserve filtrations. A morphism is said to be a quasi-isomorphism, it the induced map on the associated graded is a quasi-isomorphism. We denote the category of complete filtered vector spaces by* FVect*, and the category of complete filtered cochain complexes by* dgFVect.

For two complete filtered cochain complexes V and W we define their tensor product $V \otimes W$ by $\varprojlim (V/F^iV) \otimes (W/F^jW)$ and endow it with a natural filtration. For a collection $V_i \in$ dgFVect we define their completed direct sum by $\oplus V_i = \varinjlim(\oplus V_i/F^kV_i)$. Note that $\oplus V_i$ is the coproduct of the collection V_i. These two operations make dgFVect a symmetric monoidal category.

Consider $A \in$ Alg(dgFVect), i.e. a complete filtered cochain complex together with an associative multiplication map $A \otimes A \to A$ such that $F^iA \cdot F^jA \subset F^{i+j}A$. We are particularly interested in the following two examples.

Example 1.5. *If $\mathfrak{g}$ is a dg Lie algebra, then $CE(\mathfrak{g}) \in \mathrm{Alg}(\mathrm{dgFVect})$ with the filtration given by $F^iCE(\mathfrak{g}) = \mathrm{Sym}^{\geq i}(g^*[-1])$.*

We will use this example to show that this definition of dgFVect carries much more structure then just taking the associated graded. For a $\mathfrak{g}$-module M we set $CE(\mathfrak{g}, M) = CE(\mathfrak{g}) \otimes M$ to be a module over dg algebra $CE(\mathfrak{g})$. We have an induced filtration $F^iCE(\mathfrak{g}, M) = F^iCE(\mathfrak{g}) \otimes M$.

Lemma 1.6. *Let M, N be two $\mathfrak{g}$-modules. Then $CE(\mathfrak{g}, M)$ and $CE(\mathfrak{g}, N)$ are quasi-isomorphic as complete filtered modules over dg-algebra $CE(\mathfrak{g})$ if and only if M is quasi-isomorphic to N as $\mathfrak{g}$-modules.*

Proof. Let $f : CE(\mathfrak{g}, M) \to CE(\mathfrak{g}, N)$ be a quasi-isomorphism. By definition, we have a collection of quasi-isomorphisms $\mathrm{gr}^k CE(\mathfrak{g}, M) = \Lambda^k\mathfrak{g}^* \otimes M \to \Lambda^k\mathfrak{g}^* \otimes N = \mathrm{gr}^k CE(\mathfrak{g}, N)$. For $k = 0$ we have a quasi-isomorphism $\underline{f} : M \to N$. For $k = 1$ we have a quasi-isomorphism $\mathfrak{g}^* \otimes M \simeq \mathfrak{g}^* \otimes N$, that shows that $\underline{f}$ is compatible with a $\mathfrak{g}$-module structure.

In the opposite direction the proof is straightforward. □

Example 1.7. *For any Lie algebra the universal enveloping algebra $U(\mathfrak{g})$ has a natural filtration such that $F_iU(\mathfrak{g})$ is spanned by words of length $\leq i$ in the generators $\mathfrak{g}$. Dually, we get an increasing filtration $F^iU(\mathfrak{g})^\vee = (U(\mathfrak{g})/F_iU(\mathfrak{g}))^\vee$, and therefore $U(\mathfrak{g})^\vee \in \mathrm{Alg}(\mathrm{FVect})$.*

Now suppose that A is augmented. We can define the filtered Koszul dual of A by $A^! = \varprojlim(\mathbb{C} \otimes^{\mathbb{L}}_{A/F^iA} \mathbb{C})$. Analogously to the classical case, this derived tensor product can be computed by the bar complex, that is defined in the following way.

For a dg-algebra A we set $\overline{B}A = \oplus_n A^{\otimes n}[n]$ with the differential $d = d_A + d_m$, where $d_A(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \pm a_1 \otimes \ldots \otimes da_i \otimes \ldots \otimes a_n$ and $d_m(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \pm(-1)^i a_1 \otimes \ldots \otimes a_ia_{i+1} \otimes \ldots \otimes a_n$. The signs here are defined by Koszul rule.

Proposition 1.8. *Let $\mathfrak{g}$ be any Lie algebra. Then, $CE(\mathfrak{g})^! \simeq U(\mathfrak{g})^\vee$ as Hopf algebras.*

Proof. The filtration from Example 1.5 extends to a decreasing filtration on $CE(\mathfrak{g})^!$. To compute $\varprojlim(\oplus_n(CE(\mathfrak{g})/F^mCE(\mathfrak{g}))^{\otimes n}[n])$ we consider the spectral sequence $E^\bullet_{pq}$ of the filtration. For $q < m$ we have $E^0_{pq} = \bigoplus_{k_1+k_2+\ldots k_{-q}=p} \Lambda^{k_1}(\mathfrak{g}^*)[k_1] \otimes \ldots \otimes \Lambda^{k_{-q}}(\mathfrak{g}^*)[k_{-q}]$. Note that this spectral sequence is bounded and therefore converges $E^\infty_{pq} \to H^{p+q}((CE(\mathfrak{g})/F^mCE(\mathfrak{g}))^!)$. We have $E^0_{p\bullet}$ is the complex $\Lambda^1(\mathfrak{g}^*) \otimes \Lambda^1(\mathfrak{g}^*) \otimes \ldots \otimes \Lambda^1(\mathfrak{g}^*) \to \ldots \to \Lambda^{-q}(\mathfrak{g}^*)$, so $E^1_{pq} = S^p(\mathfrak{g}^*)$ if $q = -p$ and 0 else. All further differentials in the spectral sequence are trivial, so taking the limit we get an isomorphism of associative algebras $H^0(CE(\mathfrak{g})^!) \simeq \hat{S}(\mathfrak{g}^*)$, where the latter one stands for the completed symmetric algebra of $\mathfrak{g}^*$. We have an isomorphism of commutative algebras $U(\mathfrak{g})^\vee \to \hat{S}(\mathfrak{g}^*)$ induced from the PBW map of coalgebras $\phi : S(\mathfrak{g}) \to U(\mathfrak{g})$. Therefore $CE(\mathfrak{g})^! \simeq U(\mathfrak{g})^\vee$.

Both coproducts are induced from the coproduct in the tensor coalgebra, so this is an isomorphism of Hopf algberas. □

2. Filtered Koszul duality for FVect_h

2.1. **Generalities and the main results.** We want to show that Koszul duality from Section 1 is actually a duality. To be more precise, we have an equivalence of categories A-mod $\simeq A^!$-comod for correctly defined subcategories of modules and comodules respectively. We will prove it in a more general setting. Let FVect_h be the category of free complete filtered modules over $\mathbb{C}[[h]]$, i.e. objects in FVect_h are of the form $V \otimes \mathbb{C}[[h]]$, where $V \in \mathrm{FVect}$. Moreover, for $V \in \mathrm{FVect}_h$ we want $hF^kV \subset F^{k+1}V$. We define morphisms in FVect_h as maps of $\mathbb{C}[[h]]$-modules that respect filtration. Let $\mathcal{C} = K(\mathrm{FVect}_h)$ be the dg category of cochain complexes of objects in FVect_h, and let $\mathcal{C}^b \subset \mathbb{C}$ be a full sub dg category of objects, such that each graded piece is bounded above and below as a cochain complex.

Note that dg categories $\mathcal{C}$ and $\mathcal{C}^b$ are enriched in itself. For $V,\ W \in \mathcal{C}$ we define the filtration $F^k \operatorname{Hom}(V,W)$ on the cochain complex of morhpisms by $F^k \operatorname{Hom}(V,W) = \{f \in \operatorname{Hom}(V,W), f(F^iV) \subset F^{i+k}W$ for all $i\}$.

Then $hF^k \operatorname{Hom}(V,W) \subset F^{k+1} \operatorname{Hom}(V,W)$, and $\operatorname{Hom}(V,W) = \varprojlim \operatorname{Hom}(V,W)/F^k \operatorname{Hom}(V,W)$, so $\operatorname{Hom}(V,W) \in \mathcal{C}$.

For an algebra $A \in \operatorname{Alg}(\mathcal{C}^b)$ we always demand it to have an isomorphism $A/F^1A \simeq \mathbb{C}$. For a coalgebra $C \in \operatorname{Coalg}(\mathcal{C}^b)$ we demand an isomorphism $C/F^1C \simeq \mathbb{C}$. Let us make some definitions that will be used throughout the talk.

Definition 2.1. *Let A be as above. We say that an A-module M is free, if it is of the form $M = A \otimes V$ for an object $V \in \mathcal{C}^b$. We say that M is quasi-free, if it admits an increasing filtration $0 = G^0M \subset G^1M \subset \dots$ by sub A-modules, such that $M = \varinjlim G^iM$, and $G^iM/G^{i-1}M$ is a free A-module. We denote the dg category of quasi-free A-modules by* $A\text{-mod}$.

Definition 2.2. *Let C be as above. We say that a C-comodule M is cofree, if it is of the form $M = C \otimes V$ for an object $V \in \mathcal{C}^b$. We say that M is quasi-cofree, if it admits a decreasing filtration $M = G^0M \supset G^1M \supset \dots$ by sub A-comodules, such that $M = \varprojlim M/G^iM$, and $G^iM/G^{i+1}M$ is a cofree A-comodule. We denote the dg category of quasi-cofree A-comodules by* $C\text{-comod}$.

Definition 2.3. *We define an $A_\infty A$-module as $V \in \mathcal{C}^b$, with a collection of maps $\mu_n : A^{\otimes n} \otimes V \to V[1-n]$ for $n \geq 2$ satisfying the standard A_∞-identity, that is $\sum (-1)^{r+st} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$. Note that this is equivalent to giving the differential on $\overline{B}A \otimes V$ that is compatible with the comodule structure over the coalgebra $\overline{B}A$. We denote the dg category of $A_\infty A$-modules by $A_\infty A\text{-mod}$, where the morphisms are A_∞-maps.*

The following is the plan of this talk and simultaneously the list of the main results.

1) We construct the Koszul dual $A^! = \mathbb{C}[[h]] \otimes^{\mathbb{L}}_A \mathbb{C}[[h]]$, and a quasi-equivalence of the categories $A\text{-mod} \simeq A^!\text{-comod}$.

2) Let $\operatorname{Fin}(A^!\text{-comod})$ be the full subcategory of $A^!\text{-comod}$ consisting of $M \in A^!\text{-comod}$, such that $H^*(\operatorname{gr} M/(h))$ is of finite total dimension. Set $\operatorname{Perf}(A)$ to be the full subcategory of perfect modules in $A\text{-mod}$. Then the previous equivalence restricts to a quasi-equivalence $\operatorname{Perf}(A) \simeq \operatorname{Fin}(A^!\text{-comod})$.

3) Suppose that A is an E_2-algebra, $H^i(A^!) = 0$ if $i \neq 0$, and $H^0(A^!)$ is $\mathbb{C}[[h]]$-flat. Then there is a Hopf algebra structure on $H^0(A^!)$, and a quasi-equivalence of monoidal dg categories $A\text{-mod} \simeq H^0(A^!)\text{-comod}$.

2.2. **Quasi-free modules.** In this subsection we want to explain why we want to work with quasi-free modules. We have the following proposition.

Proposition 2.4. *Let A be an algebra in $\mathcal{C}^b$. Then, every quasi-isomorphism $f : M \to N$ of quasi-free A-modules is a homotopy equivalence. The analogous statement holds for quasi-cofree comodules.*

Proof. We will prove the proposition for quasi-free modules over an algebra. The coalgebra case is analogous. We need the following lemma.

Lemma 2.5. *Let N, M_1, M_2 be quasi-free A-modules, and $f : M_1 \to M_2$ be a quasi-isomorphism. Then the map $\operatorname{Hom}(N, M_1) \to \operatorname{Hom}(N, M_2)$ is a quasi-isomorphism.*

Proof. For any module M we have $\operatorname{Hom}(N,M) = \operatorname{Hom}(\varinjlim G^kN, M) = \varprojlim \operatorname{Hom}(G^kN, M)$. We will show that the map $\operatorname{Hom}(G^kN, M_1) \to \operatorname{Hom}(G^kN, M_2)$ is a quasi-isomorphism using induction on k. Since $G^kN/G^{k-1}N$ is free, the natural map $\operatorname{Hom}(G^kN, M) \to \operatorname{Hom}(G^{k-1}N, M)$ is surjective.

Therefore we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(G^kN/G^{k-1}N, M_1) & \longrightarrow & \mathrm{Hom}(G^kN, M_1) & \longrightarrow & \mathrm{Hom}(G^{k-1}N, M_1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(G^kN/G^{k-1}N, M_2) & \longrightarrow & \mathrm{Hom}(G^kN, M_2) & \longrightarrow & \mathrm{Hom}(G^{k-1}N, M_2) & \longrightarrow & 0 \end{array}$$

It induces a map between the long exact sequences of cohomologies, where 5-lemma gives us the desired result. ☐

To deduce the proposition we look at the quasi-isomorphism $\mathrm{Hom}(N, M) \xrightarrow{f\circ} \mathrm{Hom}(N, N)$. For $\mathrm{Id}_N \in \mathrm{Hom}(N, N)$ we have a unique up to homotopy $g \in \mathrm{Hom}(N, M)$, such that $f \circ g$ Id_N. That is a right homotopy inverse to f, and therefore quasi-isomorphism. Considering the quasi-isomorphism $\mathrm{Hom}(M, N) \xrightarrow{g\circ} \mathrm{Hom}(M, M)$ we find a right homotopy inverse to g, which must be homotopic to f. ☐

2.3. **The main property of the Koszul duality.** For a dg algebra A over $\mathbb{C}$ it is well-known that under some assumptions the categories A-mod of A-modules and $A^!$-comod of $A^!$-comodules are quasi-equivalent. In this section we will prove an analogue of this theorem for $A \in \mathrm{Alg}(\mathcal{C}^b)$.

Theorem 2.6. *There are quasi-equivalences of dg categories* $A\text{-mod} \simeq A_\infty A\text{-mod} \simeq A^!\text{-comod}$.

Proof. Let $\mathbb{1}$ stand for $\mathbb{C}[[h]]$, the unit object in $\mathcal{C}^b$. Recall that A is augmented as $\mathbb{C}[[h]]$-algebra, i.e. we have $A = \mathbb{1} \oplus \bar{A}$ for some non-unital algebra $\bar{A}$. The reduced bar complex $BA := \oplus_n \bar{A}^{\otimes n}[n] = \prod_n \bar{A}^{\otimes n}[n]$ gives a model for $A^!$. For every m, the corresponding graded piece is bounded, so $A^! \in \mathcal{C}^b$.

Let us first construct a quasi-equivalence $A_\infty A\text{-mod} \simeq A^!\text{-comod}$. For an $A_\infty A$-module N we set $N^! = A^! \otimes N = \oplus_n \bar{A}^{\otimes n}[n] \otimes N$. The $A_\infty A$-module structure on N gives a differential $d_{N^!}$ on $N^!$, so $N^!$ is a dg $A^!$-comodule. We have to check that $N^!$ is a quasi-free $A^!$-module. Since $N \in \mathcal{C}^b$, we have a decreasing filtration F^iN, and $N = \varprojlim N/F^iN$. We set $G^iN^! = A^! \otimes F^iN$ to be a decreasing filtration of $N^!$ by sub-$A^!$-comodules. By construction, $N^! = \varprojlim N^!/G^iN^!$, and $G^iN^!/G^{i+1}N^!$ is a free $A^!$-comodule. Therefore $N^! \in A^!\text{-comod}$.

To finish the proof of the second quasi-equivalence of Theorem, we want to show that every quasi-cofree $A^!$-comodule is homotopy equivalent to a one obtained as above. For $M \in A^!\text{-comod}$ we consider its free resolution of $A^!$-comodules. That is a $A^!$-comodule of the form $A^! \otimes V$ for $V \in \mathcal{C}^b$ that is quasi-isomorphic to M. It is enough to show that $A^! \otimes V$ can be obtained from an $A_\infty A$-module or, equivalently, endow V with the structure of $A_\infty A$-module. The differential d on $A^! \otimes V$ is defined by the collection of maps $d^k : \bar{A}^{\otimes k} \otimes V \to V$. This collection of maps endows V with a structure of $A_\infty A$-module.

Now we focus on the quasi-equivalence $A\text{-mod} \simeq A_\infty A\text{-mod}$. Every A-module M has a natural structure of $A_\infty A$-module. We have to show that for any quasi-free A-modules M and N the natural map $\mathrm{Hom}_{A\text{-mod}}(M, N) \to \mathrm{Hom}_{A_\infty A\text{-mod}}(M, N)$ is a quasi-isomorphism, and that any $A_\infty A$-mod is homotopy equivalent to a strict A-module.

We define the standard bar resolution $BM = \oplus_{n\geq 0} A \otimes \bar{A}^{\otimes n} \otimes M[n]$. Note that we have a natural quasi-isomorphism $BM \to M$. An argument similar the one for $N^!$ above shows that BM is a quasi-free A-module. Therefore, by Lemma 2.5 $\mathrm{Hom}_{A\text{-mod}}(BM, N) \to \mathrm{Hom}_{A\text{-mod}}(M, N)$ is a quasi-isomorphism. Note that $\mathrm{Hom}_{A\text{-mod}}(BM, N) = \mathrm{Hom}_{A_\infty A\text{-mod}}(M, N)$ by the definition of a morphism of $A_\infty A$-modules, so $\mathrm{Hom}_{A\text{-mod}}(M, N) \to \mathrm{Hom}_{A_\infty A\text{-mod}}(M, N)$ is a quasi-isomorphism.

Now let M be a $A_\infty A$-module. Then BM is a quasi-free A-module, and there is a homotopy equivalence between BM and M. ☐

2.4. **Quasi-equivalence between categories of perfect modules and finite comodules.** Let us introduce some new notations. We say that $V \in \mathcal{C}^b$ is *finite* if $H^*(\operatorname{gr} V/(h))$ is of finite total dimension. Let $M = V \otimes_{\mathbb{C}[[h]]} A$ for $V \in \mathcal{C}^b$ be a free A-module. We say that M is *of finite rank* if V is finite as an object in $\mathcal{C}^b$.

An A-module M is *perfect* if the $\operatorname{gr} A$-module $\operatorname{gr} M$ is quasi-isomorphic to the one obtained from free $\operatorname{gr} A$ modules of finite rank by a finite number of iterated cones. We denote the full sub dg-category of perfect A-modules by $\operatorname{Perf}(A)$. As in the classical case, $\operatorname{Perf}(A)$ is equivalent to the full subcategory of dualizable objects.

We say that an $A^!$-comodule N is *finite* if it is finite as an object in $\mathcal{C}^b$. We denote the full sub dg-category of finite $A^!$-comodules by $\operatorname{Fin}(A^!)$.

Proposition 2.7. *The quasi-equivalence of Theorem 2.6 restricts to a quasi-equivalence* $\operatorname{Perf}(A) \simeq \operatorname{Fin}(A^!)$.

Proof. Consider $M \in \operatorname{Perf}(A)$. We have to check that $M^! = \mathbb{1} \otimes^{\mathbb{L}}_A M$ is finite. From the construction of the filtration on $M^!$ we have $\operatorname{gr} M^! = \mathbb{1} \otimes^{\mathbb{L}}_{\operatorname{gr} A} \operatorname{gr} M$. The functor $\mathbb{1} \otimes^{\mathbb{L}}_A$ sends cones to cones, and the cone of two finite comodules is a finite comodule. Therefore it is enough to show that for $M = V \otimes_{\mathbb{C}[[h]]} A$ we have $M^! \in A^!$-comod if V is finite.

We have $\operatorname{gr} M = \operatorname{gr} A \otimes_{\mathbb{C}[h]} \operatorname{gr} V$. Therefore, $\operatorname{gr} M^!/(h) = (\mathbb{1} \otimes^{\mathbb{L}}_{\operatorname{gr} A} \operatorname{gr} M)/(h) = \operatorname{gr} V/(h)$, so $M^!$ is finite. It is easy to see that we can obtain any finite $A^!$-comodule in this way.

Now suppose that $\mathbb{1} \otimes^{\mathbb{L}}_{\mathbb{C}[[h]]} M$ is finite for some $M \in A$-mod. We want to show that M is perfect. Let us consider a free resolution of $\operatorname{gr} M$ as a $\operatorname{gr} A$-module. It gives a quasi-isomorphism $\operatorname{gr} A \otimes_{\mathbb{C}} V \simeq \operatorname{gr} M$. Note that $\operatorname{gr}(\mathbb{1} \otimes^{\mathbb{L}}_A M)/(h) = \operatorname{gr} V/(h)$, so V is finite.

Let V^i be the i-th graded component of V. Note that $V^i = 0$ for $i < 0$, and $H^*(V^i) = 0$ for $i >> 0$. The differential on $V \otimes \operatorname{gr} A$ is given by the collection of maps $d_i : V^i \to V^i \oplus \bigoplus_{0 \leq j < i} V^j \otimes \operatorname{gr}^{i-j} A$. We define the increasing filtration $F^i(V \otimes \operatorname{gr} A) = \oplus_{j \leq i} V^j \otimes \operatorname{gr} A$. Note that this is a filtration by dg $\operatorname{gr} A$ modules. Since $H^*(V^i) = 0$ for $i >> 0$, we have $F^k(V \otimes \operatorname{gr} A) = V \otimes \operatorname{gr} A$ for k big enough. Therefore we have a finite filtration of $\operatorname{gr} M$, such that the associated graded parts are free $\operatorname{gr} A$-modules. So M is perfect. □

3. Koszul dual of E_2 algebra is a Hopf algebra

3.1. **Equivalence between** C-comod **and** $H^0(C)$-comod**.** The main goal of this section is to prove the following proposition.

Proposition 3.1. *Let $C \in \operatorname{Coalg}(\mathcal{C}^b)$ be a coalgebra, such that $H^i(C) = 0$ for $i \neq 0$, and $H^0(C)$ is a flat $\mathbb{C}[[h]]$-module. Then there is a equivalence of dg categories* C-comod $\simeq H^0(C)$-comod.

Proof. The proof consists of two steps. We will first produce a coalgebra $D \in \operatorname{Coalg}(\mathcal{C}^b)$, and coalgebra maps $C \to D$ and $H^0(C) \to D$ that are quasi-isomorphisms. After that we show that if a coalgebra map $C \to D$ is a quasi-isomorphism, then there is a quasi-equivalence C-comod $\to$ D-comod.

Let d stand for the differential in C, and C^i for the piece in i-th cohomological degree. We define a cochain complex D by $D^i = C^i$ for $i > 0$, $D^i = 0$ for $i < 0$, and $D^0 = C^0/\operatorname{Im} d$. We have a natural map $C \to D$. Note that the kernel is a coideal in C, so this map endows D with a correctly defined coalgebra structure. Since $H^0(C)$ is $\mathbb{C}[[h]]$-flat, D is $\mathbb{C}[[h]]$-flat, so $D \in \operatorname{Coalg}(\mathcal{C}^b)$. The natural embedding $H^0(C) \to D$ is a coalgebra map, so we are done with the first step.

Now let C, D be coalgebras in $\mathcal{C}^b$, such that $C/F^1C = \mathbb{C} = D/F^1D$, and $\Phi : C \to D$ be a quasi-isomorphism. Let $\bar{C}$ and $\bar{D}$ be the kernels of the corresponding counit maps. We define the functors $\Phi_* : C$-comod $\to D$-comod and $\Phi^* : D$-comod $\to C$-comod in the following way.

Recall that for a left comodule M and a right comodule N over a coalgebra C we can define the cotensor product $M \odot_C N$ in the following way. Let Δ_M and Δ_N be the comodule structures.

Then we have two maps $\Delta_M \otimes \mathrm{id}$, $\mathrm{id} \otimes \Delta_N : M \otimes N \to M \otimes C \otimes N$. We set $M \odot_C N$ to be the equalizer.

For $M \in C$-comod we set $\Phi_*(M) = M \odot_D^{\mathbb{R}} D = \prod (M \otimes \bar{D}^{\otimes n} \otimes D)[-n]$ to be the cobar resolution of M as a D-comodule. As for the bar construction, the differential here is the sum of two pieces. The first one is applying the differentials of M and D to every multiplier. The second one is using the coalgebra structure.

Dually, for $N \in D$-comod we set $\Phi^*(N) = N \odot_D^{\mathbb{R}} C$.

We want to show that there are natural quasi-isomorphisms $\mathrm{Id}_{C\text{-comod}} \simeq \Phi^*\Phi_*$ and $\Phi_*\Phi^* \simeq \mathrm{Id}_{D\text{-comod}}$. We will construct the first one, the second is analogous.

Let M be a C-comodule. We have natural quasi-isomorphisms $M \simeq M \odot_C^{\mathbb{R}} C$ as C-comodules and $M \simeq \Phi_*(M) = M \odot_D^{\mathbb{R}} D$ as D-comodules. Applying the functor Φ^* we get a quasi-isomorphism $M \odot_D^{\mathbb{R}} C \simeq \Phi^*\Phi_* M$. It remains to construct a quasi-isomorphism $M \odot_C^{\mathbb{R}} C \to M \odot_D^{\mathbb{R}} C$.

Let $M \odot_C^{\triangle} C$ be a cosimplicial object, whose n-simplices are $M \otimes \bar{C}^{\otimes n} \odot C$, and face and degeneracy morphisms are the classical maps on cobar construction. Note that $M \odot_C^{\mathbb{R}} C$ is the totalization of $M \odot_C^{\triangle} C$. Similarly, let $M \odot_D^{\triangle} C$ be a cosimplicial object, whose n-simplices are $M \otimes \bar{D}^{\otimes n} \odot C$. The quasi-isomorphism $\Phi : C \to D$ induces a map of cosimplicial complexes $M \odot_C^{\triangle} C \to M \odot_D^{\triangle} C$ that is a levelwise quasi-isomorphism. Therefore the map on totalizations is a quasi-isomorphism. □

3.2. **Koszul dual of E_2-algebra.** In this section we will prove the main theorem of the talk. We need the following proposition.

Proposition 3.2. *Let $A \in \mathrm{Alg}(\mathcal{C}^b)$ be an augmented E_2-algebra. Then the category A-mod is monoidal up to homotopy.*

Proof. In this proof we follow [CG16, Section 8.2]. We aim to explain, why this fact should be true and how the proof should look like instead of giving the formal proof, so we omit details about ∞-issues that arise here.

Let Cat be the $(\infty, 2)$-category of all ∞-categories. For any monoidal category C we have a full subcategory $\mathrm{Alg}_{E_1}(C)$ of algebra objects up to homotopy. We have a functor $\mathrm{LMod}_\bullet : \mathrm{Alg}_{E_1}(C) \to \mathrm{LMod}_C(\mathrm{Cat})$ that sends an E_1-algebra $A \in C$ in the category of left modules over A.

We have the following proposition due to Lurie.

Proposition 3.3. *[Lur] $\mathrm{Alg}_{E_n}(C) \simeq \mathrm{Alg}_{E_{n-1}}(\mathrm{Alg}_{E_1}(C))$.*

Let us apply the functor Alg_{E_1} to the functor LMod. Then we get a functor $\mathrm{LMod}^2_\bullet : \mathrm{Alg}_{E_2}(C) \to \mathrm{Alg}_{E_1}(\mathrm{LMod}_C(\mathrm{Cat}))$. We have a natural forgetful functor $\mathrm{Alg}_{E_2}(C) \to \mathrm{Alg}_{E_1}(C)$. It induces the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Alg}_{E_2}(C) & \xrightarrow{\mathrm{Alg}_{E_1}(\mathrm{LMod}_\bullet)} & \mathrm{Alg}_{E_1}(\mathrm{LMod}_C(\mathrm{Cat})) \\
\downarrow & & \downarrow \\
\mathrm{Alg}_{E_1}(C) & \xrightarrow{\mathrm{LMod}_\bullet} & \mathrm{LMod}_C(\mathrm{Cat})
\end{array}$$

Therefore the category $\mathrm{LMod}(A)$ of the left A-modules has a monoidal structure (up to homotopy) coming from the multiplication in $\mathrm{Alg}_{E_1}(\mathrm{LMod}_\bullet)(A)$. □

Suppose that $H^i(A^!) = 0$ for $i \neq 0$, and $H^0(A^!)$ is $\mathbb{C}[[h]]$-flat. Then by Theorem 2.6 and Proposition 3.1 we have a quasi-equivalence A-mod $\simeq H^0(A^!)$-comod, so the former one induces a monoidal structure from the one on A-mod. We want to show that it induces a Hopf algebra structure on $H^0(A^!)$.

Theorem 3.4. *Let A be as above. Then $H^0(A^!)$ is a Hopf algebra.*

We need the standard Tannaka-Krein formalism to deduce the theorem. Let us recall its framework following [Eti+16].

Let $\mathcal{C}$ be a monoidal category. Let Vect be the monidal category of vector spaces. Suppose that we have a fiber functor $F : \mathcal{C} \to \mathrm{Vect}$, i.e. an exact faithfull functor endowed with a natural isomorphism $J_{X,Y} : F(X) \otimes F(Y) \simeq F(X \otimes Y)$ for every pair X, Y of objects in $\mathcal{C}$.

For the pair $(\mathcal{C}, F)$ we can define a coalgebra $\mathrm{Coend}(F) := (\oplus_{X\in\mathcal{C}} F(X)^* \otimes F(X))/E$, where E is generated by elements of the form $y^* \otimes F(f)x - F(f)y^* \otimes x$ for $x \in F(X)$, $y^* \in F(Y)^*$ and $f \in \mathrm{Hom}(X, Y)$. If $\mathcal{C}$ is monoidal, $\mathrm{Coend}(F)$ has an algebra structure. Indeed, $\mathrm{Coend}(F) = \mathrm{End}(F)^\vee$. On $\mathrm{End}(F)$ we have a coalgebra structure given in the following way.

We have a functor $F \times F : \mathcal{C} \times \mathcal{C} \to \mathrm{Vect}$, and a universal category $\mathcal{C} \boxtimes \mathcal{C}$, such that the product of any two right exact functors $\mathcal{C} \to \mathrm{Vect}$ uniquely factors through $\mathcal{C} \boxtimes \mathcal{C}$. Let $F \boxtimes F : \mathcal{C} \boxtimes \mathcal{C} \to \mathrm{Vect}$ be the corresponding functor. Note that we have $\mathrm{End}(F \boxtimes F) = \mathrm{End}(F) \otimes \mathrm{End}(F)$. We set $\Delta(a)_{X,Y} = J^{-1}_{X,Y} a_{X\otimes Y} J_{X,Y}$. That is a correctly defined coproduct, that induces an algebra structure on $\mathrm{Coend}(F)$. Moreover, if $\mathcal{C}$ is rigid, then $\mathrm{Coend}(F)$ admits an antipode.

Now let H be a Hopf algebra. We define the category H-comod of left H-comodules. It has a monoidal structure coming from the product on H. The natural forgetful functor to Vect is a fiber functor.

Theorem 3.5. *[Eti+16, Theorem 5.4.1]*
The assignments $(\mathcal{C}, F) \to H = \mathrm{Coend}(F)$, $H \to (H\text{-comod}, \mathrm{Forget})$ *are mutually inverse bijections between the set of rigid monoidal categories with a fiber functor* F*, up to monoidal equivalence and isomorphisms of monoidal functors, and the set of Hopf algebras up to isomorphism.*

Proof of Theorem 3.4. We will use this formalism in slightly bigger generality. Composing Section 1 and Proposition 3.1 we get the quasi-equivalence $A\text{-mod} \simeq H^0(A^!)\text{-comod}$ that induces a monoidal up to homotopy structure on $H^0(A^!)\text{-comod}$. Let $H^0(A^!)\text{-comod}^0$ be the category of free non-dg $H^0(A^!)$-modules. Since any quasi-free module is a limit of finite extensions of free modules, the monoidal structure on $H^0(A^!)\text{-comod}$ can be constructed from the one on $H^0(A^!)\text{-comod}^0$. Costello's idea is to show that this monoidal structure coincides with the original one, so the monoidal up to homotopy structure on $H^0(A^!)\text{-comod}$ is actually a strict monoidal structure.

Since $A \to \mathbb{C}[[h]]$ is a map of E_2-algebras, the corresponding forgetful functor $A\text{-mod} \to \mathcal{C}^b$ is a monoidal functor. Therefore we have an exact faithful monoidal functor $H^0(A^!)\text{-comod} \to \mathcal{C}^b$. Applying Theorem 3.5 we get a bialgebra structure on $H^0(A^!)$. Restricting additionally to the category $\mathrm{Fin}(H^0(A^!)) \simeq \mathrm{Perf}(A)$, we have a left dual for every object, so $H^0(A^!)$ admits an antipode.
□

References

[CG16] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory.* Cambridge University Press, 2016. DOI: 10.1017/9781316678626. URL: https://doi.org/10.1017/9781316678626.

[Cos13] Kevin Costello. "Supersymmetric gauge theory and the Yangian". In: (Mar. 2013). URL: https://arxiv.org/abs/1303.2632.

[Eti+16] Pavel Etingof et al. *Tensor Categories (Mathematical Surveys and Monographs).* Amer Mathematical Society, 2016. ISBN: 1470434415. URL: http://www-math.mit.edu/~etingof/egnobookfinal.pdf.

[Lur] Jacob Lurie. *Higher Algebra.* URL: http://www.math.harvard.edu/~lurie/papers/HA.pdf.

Department of Mathematics, Northeastern University. Boston, MA 02115. USA.
Email address: matvieievskyi.d@husky.neu.edu