Quantum groups

Groups and commutative/cocommutative Hopf algebras

Let's start with basics of quantum groups. Suppose G is a group. Then the space of functions on G – $Fun(G)$ (rightly understood) is obviously a unital associative algebra. But it also carries and additional structure given by duals of multiplication and inverse maps of G. They are given by $m^* : Fun(G) \to Fun(G \times G) = Fun(G) \otimes Fun(G)$ and $s^* : Fun(G) \to Fun(G)$. The dual of inclusion of identity also gives $1^* : Fun(G) \to Fun(pt)$. Associativity of $m*$ and 1 being identity leads to m^*, 1^* giving a structure of counital coassociative algebra. These maps also are by contruction algebra homomorphisms, so what the get is a bialgebra. To account for the last couple of identies concerning i^* we define what the Hopf algebra is:

Definition. A hopf algebra H is unital associative counital coassociative algebra, such that counit and comupliplication (denoted ε and Δ) are algebra morphisms with the following additional structure. The map $S : H \to H$ such that $\mu \circ (1 \otimes S) \circ \delta = \mu \circ (S \otimes 1) \circ \delta = i \circ \varepsilon$.

It also follows easily that μ and i are coalgebra maps, S is algebra morphism and coalgebra antihomomorphism, and $S \circ i = i$, $\varepsilon \circ S = \varepsilon$.

So it follows that $Fun(G)$ is a commutative Hopf algebra. By taking $Spec$ we can reverse this construction to construct a group out of a finitely generated Hopf algebra.

The second example of a Hopf algebra is an universal enveloping algbera of $\mathfrak{g}$. Here the Hopf algebra structure is given by $\Delta(x) = 1 \otimes x + x \otimes 1$ on $x \in \mathfrak{g}$ and is extended by linearity on the whole algebra. Also $S(x) = -x$. This Hopf algebra is in term cocommutative. (And is dual to the previous construction if we think about $U(\mathfrak{g})^*$ as functions on the formal group)

We also have the similar inverse result.

Proposition. *If A is cocommutative Hopf algebra generated by primitive elements ($\Delta(x) = 1 \otimes x + x \otimes 1$, then A is a universal enveloping of some Lie algebra.*

Proof. Let $\mathfrak{g}$ be the set of primitive elements, since:

$$\Delta([x, y]) = [\Delta(x), \Delta(y)] = [x, y] \otimes 1 + 1 \otimes [x, y] \ ,$$

it follows that $\mathfrak{g}$ is a Lie algebra. Since it generates A we get a surjective map $U(\mathfrak{g}) \to A$ with kernel I. This is a Hopf Ideal with a filtration induced by filtration of $U(\mathfrak{g})$ – $I = \Cup I_i$. We have $I_0 = I_1 = 0$ by definition. Suppose $x \in I_k$ with minimal k. Then $\Delta'(x) = \Delta(x) - 1 \otimes x - x \otimes 1 \in I_{k-1} \otimes I_{k-1}$, by minimality $\Delta'(x) = 0$, but then x is primitive, which is a contradiction. □

Quantization

Suppose we have A – associative algebra topologically free over $k[[h]]$, such that $A_0 = A/hA$ is also commutative. Then we also get another structure on A_0. Consider $[a], [b] \in A_0$, we have $[a, b]$ zero modulo h, so we can divide by h. This element modulo h depends only on classes of a, b:

$$\frac{1}{h}[a + ch, b] \ mod \ h = \frac{1}{h}[a, b] \ mod \ h + [c, b] \ mod \ h = \frac{1}{h}[a, b] \ mod \ h \ .$$

So this defines a operation $\{\cdot, \cdot\}$ on A_0, which satisfies:

$$\{ab, c\} = \{a, b\}c + a\{b, c\} \ .$$

So A_0 is a Poisson algebra. So we give the following definition:

Definition. A is a quantization of Poisson algebra A_0 if A is an associative algebra topologically free over $k[[h]]$, and $A_0 = A/hA$ such that the Poisson structure on A_0 and the Poisson structure described above are the same.

If one works with Hopf algebras instead of associative algebras, this definition can easily be extended to account for the quantizations of Poisson-Hopf algebras.

These on the other hand are objects dual to Poisson-Lie groups, i.e. a group with a Poisson bi-vector field compatible with group structure.

Now one can see that the Lie algbera of Poisson-Lie group carries an additional structure given by the differential of Poisson bivector field at identity (viewed as a map $G \to \Lambda^2\mathfrak{g}$). This structure is the structure of Lie bialgebra defined below.

So one can sence that if we want to work with deformations of cocommutative Hopf algebras $U(\mathfrak{g})$ it is a probably a good idea to first understand what the Lie bialgebra structure is.

Lie bialgebras

Definition. Lie bialgebra $\mathfrak{g}$ is a Lie algebra, equipped with a map $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ which defines are structure of Lie coalgebra and δ is 1-cocycle of $\mathfrak{g}$ with coefficients in $\Lambda^2\mathfrak{g}$. More explicetly:

$$((123) + (231) + (312))(\delta \otimes Id)\delta(x) = 0\ , \quad \delta([a, b]) = [\delta(a), b] + [a, \delta(b)]\ .$$

Remark 1. Here and onwards notation $[x, y]$, where $x \in \mathfrak{g}$ and y is an element of tensor algebra of $\mathfrak{g}$ means taking the corresponding adjoint action of x.

Recall that differential on $Hom(\Lambda^n\mathfrak{g}, V)$ which computes Lie algebra cohomology of $\mathfrak{g}$ with coeeficients in V is given by:

$$\partial f(x_1 \wedge \cdots \wedge x_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}x_i f(x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_{n+1}) +$$

$$+ \sum_{i<j}(-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_{n+1})\ .$$

Since δ is a cocycle it makes sence to consider the case then $\delta = \partial r$.

Definition. Cochain $r \in \Lambda^2\mathfrak{g}$ gives a coboundry structure of $\mathfrak{g}$ if $\delta = \partial r$, i.e. $\delta(a) = [1 \otimes a + a \otimes 1, r]$. A coboundry Lie bialgebra is a Lie bialgebra with a fixed coboundry structure.

Note that coboundry structures differ by elements of $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$.

Now we want to answer the following question, which elements $r \in \Lambda^2\mathfrak{g}$ give rise to a Lie bialgebra structure on $\mathfrak{g}$. We know that ∂r satisfies the cocycle condition, so we only need to understand what is the counterpart of coJacobi identity for r.

Theorem. *Let $\mathfrak{g}$ bie a Lie algebra and $r \in \Lambda^2\mathfrak{g}$. Then ∂r gives rise to a Lie bialgebra structure on $\mathfrak{g}$ iff*

$$CYB(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \mathfrak{g}^{\otimes 3}$$

is $\mathfrak{g}$-invariant.

Proof. We want to show that:

$$Alt(\delta \otimes Id)\delta(x) + [x, CYB(r)] = 0\ ,$$

then it follows that $(\forall x \;\; [x, CYB(r)]0) \Leftrightarrow (Alt(\delta\otimes Id)\delta(x)=0)$.

In order to do this we write $r = s_i \otimes t_i$ (summation over latin symbols is suppressed), with $s_i\otimes t_i = -t_i \otimes s_i$. Then:

$$[x, CYB(r)] = [x, [s_i, s_j]\otimes t_i \otimes t_j] + [x, s_i \otimes [t_i, s_j] \otimes t_j] + [x, s_i \otimes s_j \otimes [t_i, t_j]] \ .$$

On the other hand:

$$(\delta\otimes Id)\delta(x) = (\delta \otimes Id)[x, s_i\otimes t_i] = [[x, s_i], s_j \otimes t_j]\otimes t_i + [s_i, s_j\otimes t_j]\otimes[x, t_i] \ ,$$

next one can explicetly check that the sum is actually zero. □

Definition. The coboundry Lie bialgebra $\mathfrak{g}$ is called triangular iff $CYB(r) = 0$.

We may also speak of quasi-triangular Lie bialgebras. This means that r is not antysimmetric, but it's symmetric part is $\mathfrak{g}$ invariant. Hence it defines the coboundary structure in the same way.

Coboundary and triangular Hopf algebras

Before moving into the disscussion on quantization of bialgebras, I want to define Hopf-aglebras counterparts for the above Lie bialgebra definitions.

First we define "almost cocommutative" coboundary Hopf algebra:

Definition. Coboundary Hopf algebra H is a Hopf algebra together with an invertible element $R \in H\otimes H$ such that the following holds. First, it gives cocommutativity constraint $\Delta^{op} = R\Delta R^{-1}$. Second it has "zero symmetric part" $RR_{21} = 1$. And third $R_{12}(\Delta\otimes 1)R = R_{23}(1\otimes\Delta)R$.

We also define the triangular Hopf algebra:

Definition. Triangular Hopf algebra H is a Hopf algebra with $R\in H\otimes H$ satisfying first two condition of the above definition and the hexagon relations $1(\otimes\Delta)R_{13}R_{12}$, $(\Delta\otimes 1)R = R_{13}R_{23}$.

One can see that from the third relation it follows that:

$$R_{12}R_{13}R_{23} = R_{12}(\Delta\otimes 1)R = (\Delta^{op}\otimes 1)(R)R_{12} = R_{23}R_{13}R_{12} \ .$$

So R satisfies quantum Yang-Baxter equation. From this one can also see that triangular Hopf algebra is a coboundary Hopf algebra.

Here we can also speak of quasi-triangular Hopf algebras, dropping the second assumption on R.

Quantized universal enveloping algebras

Here we want to show that if our Hopf algebra is a quantization of Universal enveloping algebra of Lie bialgebra then the above definitions indeed correspond in semi-classical limit to the Lie bialgebra definitions.

We will start with the the following definition:

Definition. The Hopf algebra H topologically free over $k[[h]]$ is called quantized universal enveloping algebra, if $H/hH = U(\mathfrak{g})$ for some $\mathfrak{g}$ as a Hopf algebra.

Proposition. *If H is a QUE algebr, then $\mathfrak{g}$ obtains a bialgebra structure given by:*

$$\delta(x) = \frac{1}{h}(\Delta(X) - \Delta^{op}(X)) \ (mod \ h) \ ,$$

for arbitary lifting X of x to H.

Proof. This definition makes sence for the same reason as Poisson structure in the commutative case. Because Δ is cocommutative to the first order in h.

So we need to show that $\delta(x) \in \Lambda^2\mathfrak{g}$ and that it satisfied cocycle and coJacobi identities.

First of all δ is skewsymmetric, so it is enough to show that $\delta(x) \in \mathfrak{g} \otimes \mathfrak{g}$. Indeed:

$$(\delta_U \otimes 1)\delta(x) = \frac{1}{h}(\delta \otimes 1)(\delta(X) - \delta^{op}(X)) \ (mod\, h) = \sum (X' \otimes X'' \otimes X''' - X'' \otimes X''' \otimes X') \ (mod\, h) =$$

$$= \sum (X' \otimes X'' \otimes X''' - X' \otimes X''' \otimes X'' + X' \otimes X''' \otimes X'' - X'' \otimes X''' \otimes X') \ (mod\ h) =$$

$$= [(1 \otimes \Delta - 1 \otimes \Delta^{op})\Delta(x) + \sigma_{23}(\Delta \otimes 1 - 1 \otimes \Delta^{op})\Delta(x)] \ (mod\ h) =$$

$$= (1 \otimes \delta)\Delta_U(x) + \sigma_{23}(\delta \otimes 1)\Delta_U(x) = 1 \otimes \delta(x) + \sigma_{23}(\delta(x) \otimes 1)\ .$$

So the $\delta(x)$ is primitive in both arguments.

Now for the coJacoby identity:

$$(\delta \otimes 1) \circ \delta(x) = \frac{1}{h^2}[(\Delta \otimes 1) \circ \Delta - (\Delta \otimes 1) \circ \Delta^{op} - (\Delta^{op} \otimes 1) \circ \Delta + (\Delta^{op} \otimes 1) \circ \Delta^{op}](X) \ (mod\ h) =$$

$$= \sum [X' \otimes X'' \otimes X''' - X'' \otimes X''' \otimes X' - X'' \otimes X' \otimes X''' + X''' \otimes X'' \otimes X'\ ,$$

and one can see that after summing all alterated versions of this up we get 0.

The last thing is cocycle condition:

$$\delta([x, y]) = \frac{1}{h}([\Delta(X), \Delta(Y)] - [\Delta^{op}(X), \Delta^{op}(Y)]) \ (mod\ h) =$$

$$\frac{1}{h}([\Delta(X), \Delta(Y)] - [\Delta^{op}(X), \Delta(Y)] + [\Delta^{op}(X), \Delta(Y)] - [\Delta^{op}(X), \Delta^{op}(Y)]] \ (mod\ h) =$$

$$= [\delta(x), \Delta_U(y)] + [\Delta_U(X), \delta(y)]\ .$$

□

Of course lurking behind the proof is a fact that the the deformation of cocommutative Hopf algebra gives it a structure of coPoisson algebra. It just happens that this coPoisson structure also descends on $\mathfrak{g}$ and gives it Lie bialgebra structure.

Now suppose that H also has a coboundary structure:

Proposition. *If H is a QUE, with coboundary structure R, then $r \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined by $R = 1 + hr$ $(mod\ h^2)$, in fact lies in $\Lambda^2\mathfrak{g}$ and gives a coboundary structure on $\mathfrak{g}$.*

Proof. First since H/hH is cocommutative we have $R = 1$ $(mod\ h)$, so definition makes sense.

Now let's check $r \in \mathfrak{g} \otimes \mathfrak{g}$. So from $R_{12}(\Delta \otimes 1)R = R_{23}(1 \otimes \Delta)R$ we get:

$$r_{12} + (\Delta_U \otimes 1)r = r_{23} + (1 \otimes \Delta_U)r\ .$$

From here one proceeds by writing down $r = \sum r_{\lambda\mu} x_\lambda \otimes x_\mu$ with $x_\lambda = \prod \frac{x_i^{\lambda_i}}{\lambda_i!}$ – being the PBW basis of $U(\mathfrak{g})$. After such an expansion the above equation translates into $r \in \mathfrak{g} \otimes \mathfrak{g}$.

Now from $RR_{21} = 1$ it follows that $r + r_{21} = 0$, so r is indeed skew. Now we only need to show $\delta = \partial r$. Indeed:

$$\delta(x) = \frac{1}{h}(\Delta(X) - R\Delta(X)R^{-1}) \ (mod\ h) = -r\Delta_U(x) + \Delta_U(x)r = [x, r]\ .$$

□

The last proposition is about triangular Hopf algebras:

Proposition. *If H is a triangular Hopf algebra, then corresponding r satisfies CYBE.*

Proof. Expand $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ to get:

$$r_{12}r_{13} + r_{13}r_{23} + r_{12}r_{23} = r_{23}r_{13} + r_{23}r_{12} + r_{13}r_{12}\ .$$

□

Example of $\mathfrak{sl}_2$

Let's see how the above constructions work out in the case of $\mathfrak{sl}_2$.

First of all one can easily classify Lie bialgebra structures on $\mathfrak{sl}_2$. It turns out there are three possiblities: trivial structure $\delta = 0$, "standard" structure

$$\delta(H) = 0 \ , \ \delta(E) = E \wedge H \ , \ \delta(F) = F \wedge H \ r = E \wedge F$$

and one more non-standard structure given by $r = H \wedge E$.

One can also show that the quantization of the "standard" structure is given by:

$$[H, E] = 2E \ , \ [H, F] = -2F \ , \ [E, F] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}} \ ,$$

$$\Delta(E) = E \otimes e^{hH} + 1 \otimes E \ , \ \Delta(F) = F \otimes 1 + e^{-Hh} \otimes F \ , \ \Delta(H) = 1 \otimes H + H \otimes 1 \ ,$$

$$S(E) = -Ee^{-hH} \ , \ S(F_= - e^{hH}F \ , \ S(H) = -H \ , \varepsilon(E/H/F) = 0 \ .$$

We can easily observe that taking $K = e^{hH}$ one obtains the usual relations for $U_q(\mathfrak{sl}_2)$.

The R-matrix is given by:

$$R = \sum_{n=0}^{\infty} R_n(h) e^{\frac{1}{2}hH\otimes H} E^n \otimes F^n \ .$$

Yangian

Definitions and uniqeness

To construct the Yangian we need to consider a quantization of Lie bialgebra $\mathfrak{g}[u]$ with a following coalgebra structure:

$$\delta(f(u, v)) = (ad_{f(u)} \otimes 1 + 1 \otimes ad_{f(v)}) \frac{t}{u - v} \ ,$$

where t is a Casimir element associated with fixed invariant form on $\mathfrak{g}$ viewed as an element of $\mathfrak{g} \otimes \mathfrak{g}$.

Note that this Lie bialgebra structure is not triangular. Because $r = \frac{t}{u-v}$ is not an element of $\mathfrak{g} \otimes \mathfrak{g}[u, v]$.

However it is pseudotriangular. Indeed, if we consider $\mathfrak{g}[u]((\lambda^{-1}))$, and define translation automorphism $\tau_\lambda : f(u) \to f(u + \lambda)$ we can take:

$$r(\lambda) = (\tau_\lambda \otimes 1) \frac{t}{u - v} = \frac{t}{u + \lambda - v} = \sum_{r=0}^{\infty} (v - u)^r \lambda^{-r-1} t \ ,$$

which now belongs to $(\mathfrak{g} \otimes \mathfrak{g})[u, v]$.

Notice that this satisfies the CYBE with spectral parameter:

$$[r_{12}(z_1 - z_2), r_{13}(z_1 - z_3)] + [r_{12}(z_1 - z_2), r_{23}(z_2 - z_3)] + [r_{13}(z_1 - z_3), r_{23}(z_2 - z_3)] = 0 \ .$$

Indeed since t is Casimir element it follows that $[t, 1 \otimes x + x \otimes 1] = 0$, but multyplying CYBE by $(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)$ we have:

$$z_2([t_{12}, t_{13}] + [t_{23}, t_{13}]) + z_1([t_{12}, t_{23}] + [t_{13}, t_{23}]) + z_3([t_{13}, t_{12}] + [t_{23}, t_{12}]) = 0 \ .$$

Since $\mathfrak{g}$ is graded by degree in u, and so is $U(\mathfrak{g})$ it is reasonable to look for a homogeneous quantization of this Lie bialgebra, i.e. for a graded quantization with $\deg h = 1$

and such that the induced grading on $U(\mathfrak{g})$ is the same as one coming from the grading on $\mathfrak{g}$.

This can be constructed in the following way. Since $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ it follows that $\mathfrak{g}$ is generated by $\mathfrak{g}$ and $\mathfrak{g}u$ as a Lie algebra. Since $\delta(x)=0$ for $x\in\mathfrak{g}$ it makes sense to start by taking $\Delta_h(x)=1\otimes x+x\otimes 1$.

Now $\delta(ux)=[xu\otimes 1+1\otimes xv,\frac{t}{u-v}]=[x\otimes 1,t](\frac{u}{u-v}-\frac{v}{u-v})=[x\otimes 1,t]$. So we need to satisfy:

$$\frac{\Delta_h(ux)-\Delta_h^{op}(ux)}{h}=[x\otimes 1,t]\ .$$

This leads to the following definition($\{z_1,z_2,z_3\}=\frac{1}{24}\sum_{s\in S_3}z_{s(1)}z_{s(2)}z_{s(3)}$):

Definition. For a simple $\mathfrak{g}$ with a fixed invariant form $(,)$ and x_λ – orthonoraml basis. There is a homogeneous quantization $U_h(\mathfrak{g}[u])$ of $\mathfrak{g}[u]$. It is topologically generatd by $x,J(x)$ for $x\in\mathfrak{g}$ with following relations:

$$[x,y]=[x,y]_{\mathfrak{g}}\ ,\ \ J(ax+by)=aJ(x)+bJ(y)\ ,\ \ [x,J(y)]=J([x,y])\ ,$$

$$[J(x),J([y,z])]+[J(z),J([x,y])]+[J(y),J([z,x])]=h^2\sum_{\lambda,\mu,\nu}([x,x_\lambda],[[y,x_\mu],[z,x_\nu]])\{x_\lambda,x_\mu,x_\nu\}\ ,$$

$$[[J(x),J(y)],[z,J(w)]]+[[J(z),J(w)],[x,J(y)]]=\sum_{\lambda,\mu,\nu}([x,x_\lambda],[[y,x_\mu],[[z,w],x_\nu]])\{x_\lambda,x_\mu,J(x_\nu)\}\ .$$

The Hopf algebra structure is given by:

$$\Delta_h(x)=x\otimes 1+1\otimes x\ ,\Delta_h(J(x)=J(x)\otimes 1+1\otimes J(x)+\frac{h}{2}[x\otimes 1,t]\ ,$$

$$S_h(x)=-x\ ,\ \ S_h(J(x))=-J(x)+\frac{1}{4}cx\ ,\ \ \epsilon_h(x)=\epsilon(J(x))=0\ ,$$

where c is eigenvalue of t on the adjoint representation. Grading is given by $\deg x=0$ and $\deg J(x)=1$.

This algebra is indeed graded. Also upon the identification $[J(x)]=ux$ we see that the coproduct indeed gives the chosen Lie bialgebra structure on $\mathfrak{g}[u]$. Also all the relations are trivially satisfied in $U(\mathfrak{g}[u])$ for $h=0$.

Now we also claim that this is unique homogeneous quantization of $\mathfrak{g}[u]$. This can be checked along the following lines. We suppose that we have a quantization which differs from the given in starting with the degree h^n. Then taking the difference of two product and coproduct maps we obtain an cocycle in the cohomology of $U(\mathfrak{g}[u])$ as a Hopf algebra. For example suppose $\mu'=\mu+\tilde{\mu}h^n$ $(mod\ h^{n+1})$, then the associativity constraint gives:

$$\mu'(x,\mu'(y,z))=\mu'(\mu'(x,y),z)\Rightarrow\tilde{\mu}(x,yz)+x\tilde{\mu}(y,z)=\tilde{\mu}(xy,z)+\tilde{\mu}(x,y)z\ .$$

The cohomology of Hopf algebra is defined to be the cohomology of the total complex associated with a bicomplex $C^{p,q}=Hom_k(H^{\otimes p},H^{\otimes q})$. There horizontal and vertical differentials are some deformations of cohomological algebra differential and homological coaglebra differential.

Now in case of QUE one can prove that morphisms $\tilde{\mu}$ and $\tilde{\Delta}$ restrict to give vectore space morphisms on the level of Lie algebra. So if $H=(U(\mathfrak{h})$ they give a cocylce in the Lie bialgebra cohomology given by the total complex associated with a bicomplex $C_l^{p,q}=Hom(\Lambda^p\mathfrak{h},\Lambda^p\mathfrak{h})$ with vertical and horizontal differential given exactly by cohomological Lie algebra differential and homological Lie coalgebra differential.

So now uniqeness of quantization amounts to showing that the first cohomology group vanishes in this complex.

Now the Yangian $Y(\mathfrak{g})$ is just $U_h(\mathfrak{g}[u])/(h-1)$. There is also another definition of Yangian in terms of generators and relations which is a little bit more natural. Note that after such a substitution the Yangian obtains a filtration comming from the grading in $U_h(\mathfrak{g}[u])$ and the associated graded algebra is $U(\mathfrak{g}[u])$

Theorem. *The Yangian* $Y(\mathfrak{g})$ *is isomorphic to associative algebra with generators* $X^{\pm}_{i,r}$, $H_{i,r}$, $i=1,\ldots,n$, $r\in\mathbb{N}$ *and* n *is a rank of Cartan matrix corresponding to* $\mathfrak{g}$. *The relations are:*

$$[H_{i,r},H_{j,s}]=0\ ,\ [H_{i,o},X^{\pm}_{j,s}]=\pm d_i a_{ij}X^{\pm}_{j,s}\ ,\ [X^{+}_{i,r},X^{-}_{j,s}]=\delta_{i,j}H_{i,r+s}\ ,$$

$$[H_{i,r+1},X^{\pm}_{j,s}]-[H_{i,r},X^{\pm}_{j,s+1}]=\pm\frac{1}{2}d_i a_{ij}(H_{i,r}X^{\pm}_{j,s}+X^{\pm}_{j,s}H_{i,r})\ ,$$

$$[X^{\pm}_{i,r+1},X^{\pm}_{j,s}]-[X^{\pm}_{i,r},X^{\pm}_{j,s+1}]=\pm\frac{1}{2}d_i a_{ij}(X^{\pm}_{i,r}X^{\pm}_{j,s}+X^{\pm}_{j,s}X^{\pm}_{i,r})\ ,$$

$$\sum_{\sigma}[X^{\pm}_{i,r_{\sigma(1)}},[X^{\pm}_{i,r_{\sigma(2)}},\ldots,[X^{\pm}_{i,r_{\sigma(m)}},X^{\pm}_{j,s}]\ldots]]=0$$

for any sequence $r_1,\ldots,r_m$, *with* $m=1-a_{ij}$.

The isomorphisms between two realizations is given by:

$$\phi(H_i)=d_i^{-1}H_{i,0}\ ,\ \phi(J(H_i))=d_i^{-1}H_{i,1}+\phi(v_i)\ ,\phi(X_i^{\pm})=X^{\pm}_{i,0}\ ,\phi(J(X_i^{\pm}))=X^{\pm}_{i,1}+\phi(w_i^{\pm})\ ,$$

for some corrections v_i, $w_i^{\pm}$.

Another basic propertiy of Yangian which we are going to need is the fact that we can quantize the translation automorphism of lie algebra as follows:

$$\tau_a(x)=x\ ,\ \tau_a(J(x))=J(x)+ax\ .$$

It is also possible to classify all the finite-dimensional representations of the Yangian. It turns out that they are classified by tuples of series $\lambda_i(u)\in 1+u^{-1}\mathbb{C}[u^{-1}]$, which generalize the highest weight from the $U(\mathfrak{g})$ case, such that there are polynomials $P_i\in\mathbb{C}[u]$:

$$\frac{P_i(u+d_i)}{P_i(u)}=h_i(u)\ .$$

Rational solutions of the quantum Yang-Baxter equation

Since $\mathfrak{g}[u]$ was a pseudotriangular Lie bialgebra, it is reasonable to expect $Y(\mathfrak{g})$ to be a pseudotriangular Hopf algebra too. Which means that there is an R-matrix with a spectral parameter, which is an element of $(Y(\mathfrak{g})\otimes Y(\mathfrak{g}))[[\lambda^{-1}]]$ such that:

$$\mathcal{R}(\lambda)=1+\frac{t}{\lambda}+\sum\mathcal{R}_r\lambda^{-r-1}\ ,$$

and it satisfies the following properties:

$$(\Delta\otimes 1)\mathcal{R}(\lambda)=\mathcal{R}_{13}(\lambda)\mathcal{R}_{23}(\lambda)\ ,\ (1\otimes\Delta)\mathcal{R}(\lambda)=\mathcal{R}_{13}(\lambda)\mathcal{R}_{12}(\lambda)\ ,$$

$$(t_\lambda\otimes 1)\Delta^{op}(x)=\mathcal{R}(\lambda)((t_\lambda\otimes 1)\Delta(x))\mathcal{R}(\lambda)^{-1}\ ,\ \mathcal{R}_{21}(\lambda)=\mathcal{R}_{12}(-\lambda)^{-1}\ ,$$

and:

$$\mathcal{R}_{12}(\lambda_1-\lambda_2)\mathcal{R}_{13}(\lambda_1-\lambda_3)\mathcal{R}_{23}(\lambda_2-\lambda_3)=\mathcal{R}_{23}(\lambda_2-\lambda_3)\mathcal{R}_{13}(\lambda_1-\lambda_3)\mathcal{R}_{12}(\lambda_1-\lambda_2)\ .$$

This is indeed so, by the theorem due to Drinfeld.

It follows that this R-matrix gives us a lot of rational solutions (they are rational by another theorem due to Drinfeld) of QYBE, which we can extract by evaluation of this R-matrix on some representation.

But it turns out that the "opposite" also true, every rational solution of QYBE of a certain form comes from the universal R-matrix of Yangian.

The exact statement is as follows. Suppose we have a solution of QYBE $R_\rho(\lambda, h) \in End(V \otimes V)[[\lambda^{-1}]][[h]]$ of the form:

$$R_\rho(\lambda, h) = 1 + h(\rho \otimes \rho)\frac{t}{\lambda} + \sum h^k a_k(\lambda) \ ,$$

which is homogeneous $R_\rho(\alpha\lambda, \alpha h) = 1$. So the solution is uniquely determined by $R_\rho(\lambda) = R_\rho(\lambda, 1)$. Here ρ is a representation of $\mathfrak{g}$ on V.

It turns out that the following theorem holds:

Theorem. *For any solution of QYBE of the form $R_\rho(\lambda)$, it is possible to extend a representation ρ to the representation of $Y(\mathfrak{g}$ – $\pi : Y(\mathfrak{g}) \to End(V)$ in such a way that: $R_\rho(\lambda) = f(\lambda)(\pi \otimes \pi)\mathcal{R}(\lambda)$.*

All the rational solutions of this sort are found in the case of $\mathfrak{g} = \mathfrak{sl}_2$.

Another interesting property of the Yangians is the following one.

Let ρ be an irreducible representation of $Y(\mathfrak{g})$. Consider an algebra A_ρ generated by $t_{ij}^{(k)}$ with $1 \le i, j \le \dim(V)$ and $k \in \mathbb{Z}_{>0}$ and the relation:

$$[\rho \otimes \rho(R(u - v))]T^I(u)T^{II}(v) = T^{II}(v)T^I(u)[\rho \otimes \rho(R(u - v))] \ ,$$

where $T(u) = \sum E_{ij} \otimes t_{ij}(u)$ and $t_{ij}(u) = \delta_{ij} + \sum t_{ij}^{(k)} u^{-k}$. The coproduct $\delta(t_i j(u)) = \sum t_{ik}(u) \otimes t_{kj}(u)$ gives A_ρ the structure of Hopf algebra.

It turns out that there is an epimorhpism $\phi : A_\rho \to Y(\mathfrak{g})$ such that $\phi(T(u)) = 1 \otimes \rho\mathcal{R}(u)$. And there are central series $c(u)$, such that $\delta(c(u)) = c(u) \otimes c(u)$ and $c(u)$ generates the kernel of ϕ.

In the case of $\mathfrak{g} = \mathfrak{sl}_n$ this gives the familiar presentation of $Y(\mathfrak{sl}_n)$ given by the algebra generated by $T(u)$ with the above with $R = 1 - \frac{P}{u}$ (representation is just the fundamental representation of $\mathfrak{sl}_n$ with trivial action of $J(u)$) and the relation $qdet(T)(u) = 1$.