AN INTRODUCTION TO FACTORIZATION ALGEBRAS

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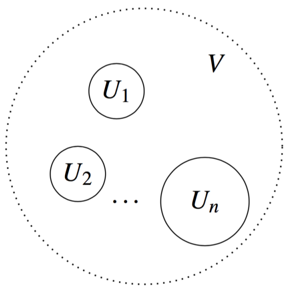
Our primary reference are the books of Costello and Gwillam [CG17, CG].

1. Prefactorization algebras

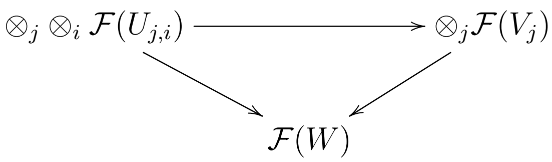
1.1. **Definitions.** A *prefactorization algebra* $\mathcal{F}$ on a topological space M, with values in $Vect^{\otimes}$ (the symmetric monodical category of vector spaces), is an assignment of a vector space $\mathcal{F}(U)$ for each open set $U \subseteq M$ together with the following data:

- For an inclusion $U \to V$, a map $\mu_U^V : \mathcal{F}(U) \to \mathcal{F}(V)$.
- For a finite collection of disjoint opens $\sqcup_{i \in I} U_i \subset V$, an $\Sigma_{|I|}$-equivariant map $\mu_{\{U_i\}}^V : \otimes_{i \in I} \mathcal{F}(U_i) \to \mathcal{F}(V)$



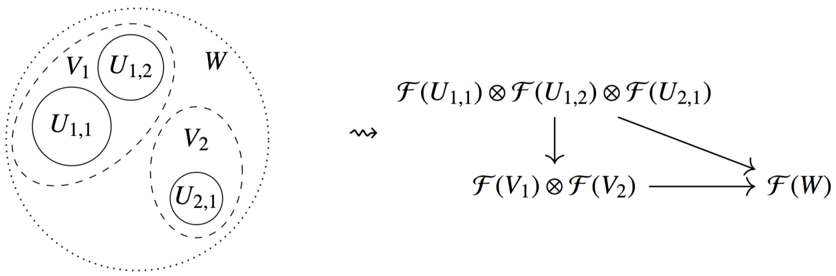
The maps $\mu_{\{U_i\}}^V$ are subject to following compatibility for sequences of inclusions:

- For a collection of disjoint opens $V_j \subset W$, and collections of disjoint opens $U_{j,i} \subset V_j$, the composite maps



  agree.

For example, the composition maps for the following configuration of three open sets are given by:



Note that $\mathcal{F}(\emptyset)$ must be a commutative algebra, and the map $\emptyset \to U$ for any open U, turns $\mathcal{F}(U)$ into a pointed vector space.

A prefactorization algebra is called *multiplicative* if

$$\otimes_i \mathcal{F}(U_i) \cong \mathcal{F}(U_1 \amalg \cdots \amalg U_n)$$

via the natural map $\mu_{\{U_i\}}^{\sqcup U_i}$.

1.1.1. *An equivalent definition.* One can reformulate the definition of a factorization algebra in the following way. It uses the following definition.

A *pseudo-tensor category* is a collection of objects $\mathcal{M}$ together with $\Sigma_{|I|}$-equivariant vector spaces

$$\mathcal{M}(\{A_i\}_{i\in I}|B)$$

for each finite open set I and objects $\{A_i, B\}$ in $\mathcal{M}$ satisfying certain associativity, equivariance, and unital axioms. Often times, pseudo-tensor categories are referred to as *colored operads*, where the colors correspond to the objects in $\mathcal{M}$.

Every linear symmetric monoidal category $\mathcal{C}^{\otimes}$ determines a pseudo-tensor category $\mathcal{M}_{\mathcal{C}}$ via the rule

$$\mathcal{M}_{\mathcal{C}}(\{A_i\}|B) = \mathrm{Hom}_{\mathcal{C}}(\otimes A_i, B).$$

Conversely, given a pseudo-tensor category $\mathcal{M}$, there is a "universal symmetric monoidal category" $\mathcal{SM}$ whose objects consist of formal expressions

$$A_1 \otimes \cdots \otimes A_n.$$

The tensor product is defined in the obvious way. The hom spaces are defined by the formula

$$\mathrm{Hom}_{\mathcal{SM}}(A_1 \otimes \cdots \otimes A_n, B) = \mathcal{M}(\{A_1, \ldots, A_n\}|B).$$

Our favorite example of a pseudo-tensor category is associated to any manifold M. The pseudo-tensor category Disj_M of *disjoint open sets* in M is given by the set of all connected open subsets of M. For every such finite collections of opens $\{U_i\}_{i\in I}$ and open V one defines

$$\mathrm{Disj}_M(\{U_i\}|V)$$

by the following rules:

- if the collection $\{U_i\}$ is pairwise disjoint and are all contained V, then $\mathrm{Disj}_M(\{U_i\}|V)$ is the set with one element;

- otherwise, $\mathrm{Disj}_M(\{U_i\}|V)$ is empty.

Composition in this pseudo-tensor category is defined in the obvious way.

One can check the following:

Proposition 1.1. *A prefactorization algebra is a symmetric monoidal functor*

$$\mathcal{F} : \mathcal{S}\mathrm{Disj}_M \to \mathrm{Vect}.$$

This motivates the following, more general definition. If $\mathcal{C}^\otimes$ is *any* symmetric monoidal category, we can consider "prefactorization algebras with values in $\mathcal{C}^\otimes$. Precisely, a prefactorization algebra on M with values in $\mathcal{C}^\otimes$ is a symmetric monoidal functor

$$\mathcal{F} : \mathcal{S}\mathrm{Disj}_M \to \mathcal{C}^\otimes.$$

Typically, for us, $\mathcal{C}^\otimes$ will be the symmetric monoidal category of vector spaces $\mathrm{Vect}^\otimes$, chain complexes $\mathrm{Ch}^\otimes$, or slight enhancements that we will discuss later on.

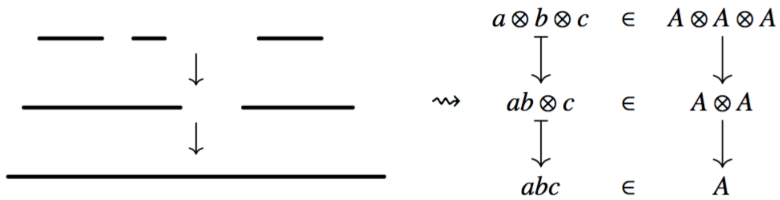
Remark 1.2.
1. Note that $\emptyset$ is a connected open subset. Thus, for each prefactorization algebra we have an object $\mathcal{F}(\emptyset) \in \mathcal{C}$. According to the definitions, $\mathcal{F}(\emptyset)$ is a commutative algebra object in $\mathcal{C}$.
2. One says that $\mathcal{F}$ is *unital* if $\mathcal{F}(\emptyset)$ is a unital commutative algebra.
3. In the language of colored operads, a prefactorization algebra is an algebra over the colored operad of disjoint open sets in M, Disj_M.
4. The category of prefactorization algebras themselves form a pseudo-tensor category $\mathrm{PreFact}_M$. If $\{\mathcal{F}_i\}$ is a finite collection of prefactorization algebras on M, and $\mathcal{G}$ is another prefactorization algebra then one defines
$$\mathrm{PreFact}_M(\{\mathcal{F}_i\}|\mathcal{G}) = \mathrm{Hom}(\otimes\mathcal{F}_i, \mathcal{G}).$$
Here, the tensor product is defined by $(\otimes\mathcal{F}_i)(U) = \otimes\mathcal{F}_i(U)$. A homomorphism of prefactorization algebras $\phi : \mathcal{F} \to \mathcal{G}$ is defined by the data: for each $U \subset M$ open, a map $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ subject to obvious compatibilities with the structure multiplication maps.

1.2. **Examples.**

1.2.1. *Associative Algebras.* The simplest examples of prefactorization algebras we give are on $\mathbb{R}$, and are built out of Associative algebras. There is a map

$$\mathrm{AssocAlg} \to \mathrm{PreFact}(\mathbb{R}) : A \to A^{fact} \tag{1}$$

The multiplicative prefactorization algebra A^{fact} assigns a copy of the associative algebra A to each open interval, $A^{fact}((a, b)) = A$. The structure maps are all given by the appropriate multiplication maps in the algebra. For example



The algebras we construct this way are *locally constant*, meaning that the maps $\mathcal{F}((a,b)) \to \mathcal{F}((c,d))$ are isomorphisms for an inclusion of intervals $(a,b) \subset (c,d)$. In fact, any locally constant prefactorization algebra on $\mathbb{R}$ recovers an associative algebra $A_{\mathcal{F}} = \mathcal{F}(\mathbb{R})$, as the local constant property guarantees that $\mathcal{F}((a,b)) \to \mathcal{F}(\mathbb{R})$ is an isomorphism for each interval. In this way, we can produced an equivalence of categories

$$\text{AssocAlg} \cong \text{PreFact}_{l.c.}(\mathbb{R}) \tag{2}$$

Here we still taking prefactorization algebras valued in vector spaces.

1.2.2. *Bimodules as domain walls (defects).* In quantum field theories, we often consider *defects*. These are a certain class of operators that are attached to submanifolds $N \subset M$. For codimension 1 defects, known as domain walls, the operator acts to separate two different QFTs on either side of the wall. We expect to find that the domain wall is a bimodule for the algebra of local observables on both sides, in a compatible way.

This setup is very naturally described by bimodules for associative algebras. Let A, B be associative algebras. For a point $p \in \mathbb{R}$ and M and $A - B$ bimodule, we construct a prefactorization algebra $\mathcal{F}_{A,M,B}$ on $\mathbb{R}$ as follows.

- For an interval (a_1, a_2) with $a_1 < a_2 < p$, we assign

$$\mathcal{F}_{A,M,B}((a_1, a_2)) = A. \tag{3}$$

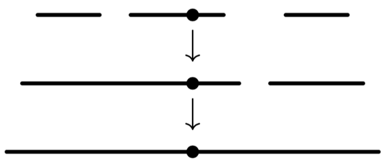
- For an interval (b_1, b_2) with $p < b_1 < b_2$, we assign

$$\mathcal{F}_{A,M,B}((b_1, b_2)) = B. \tag{4}$$

- For an interval (a, b) with $a < p < b$, we assign

$$\mathcal{F}_{A,M,B}((a, b)) = M. \tag{5}$$

For the structure maps, we have the module action $a \otimes m \to am$, etc.



$$\rightsquigarrow \quad \begin{array}{ccc} a\otimes m\otimes b & \in & A\otimes M\otimes B \\ \downarrow & & \downarrow \\ am\otimes b & \in & M\otimes B \\ \downarrow & & \downarrow \\ amb & \in & M \end{array}$$

The last structure map map we need to define is for the map $\emptyset \to (a,b)$, where $p \in (a,b)$. For this, there is no canonical choice, and we must prescribe an element $m_{(a,b)} \in M$ to each such interval.

Similarly, we may consider two defects: an $A-B$ bimodule M attached at $p \in \mathbb{R}$, and a $B-C$ bimodule N at $q \in \mathbb{R}$. We then can construct a factorization algebra $\mathcal{F}_{A,M,B,N,C}$ with this data, where the only new piece of information we need to prescribe is for intervals (a,c) that contain both p and q. The structure maps tell us that $V = \mathcal{F}_{A,M,B,N,C}((a,c))$ must receive maps from $M \otimes N$. Furthermore this map must factor through the internal B action, and thus there is a map $M \otimes_B N \to V$, but this map may not be surjective. We come back to this construction in the last section of these notes.

In physics language, this is known as *fusion* of domain walls.

1.2.3. *Enveloping Algebras.* For more interesting examples, we consider factorization algebras on $\mathbb{R}$ with values in in the category $dgVect^{\otimes}$ of differential graded vector spaces, where we invert weak equivalences.

Recall the definition of the Chevally-Eilenberg chain complex, for a Lie algebra $\mathfrak{h}$. This is a model for Lie algebra homology. Define

(6) $$C_*(\mathfrak{h}) := \mathrm{Sym}(\mathfrak{h}[1]) = \oplus_{n\geq 0} \wedge^n \mathfrak{h}[n]$$

equipped with the following degree 1 differential

(7) $$d_{CE} := [\,,] : \wedge^2\mathfrak{h}[2] \to \wedge^1\mathfrak{h}[1]$$

extended to all of $\oplus_{n\geq 0} \wedge^n \mathfrak{h}[n]$ as a derivation. The Jacobi identity for the bracket ensures that $d_{CE}^2 = 0$. From this, we can show that $C_*(\mathfrak{h}) \otimes_{\mathbb{K}} U\mathfrak{h}$ is a free resolution of $\mathbb{K}$ as a $U\mathfrak{h}$ module, where the differential is

(8) $$h_1 \wedge \cdots \wedge h_n \otimes x \mapsto d_{CE}(h_1 \wedge \cdots \wedge h_n) \otimes x + \sum (-1)^{i-n} h_1 \wedge \cdots \wedge \hat{h}_i \wedge \cdots \wedge h_n \otimes (h_i x - x h_i).$$

That is, the complex

(9) $$\cdots \wedge^2 \mathfrak{h} \otimes U\mathfrak{h} \to \mathfrak{h} \otimes U\mathfrak{h} \to U\mathfrak{h} \to \mathbb{K} \to 0$$

gives us the desired free resolution of $\mathbb{K}$. For an $\mathfrak{h}$ module M, the Lie algebra homology is defined as

(10) $$H_*(\mathfrak{h}, M) := \mathbb{K} \otimes^{\mathbb{L}}_{U\mathfrak{h}} M \cong (C_*(\mathfrak{h}) \otimes_{\mathbb{K}} U\mathfrak{h}) \otimes_{U\mathfrak{h}} M = C_*(\mathfrak{h}) \otimes_{\mathbb{K}} M.$$

This construction can easily be extended to a dgla $\mathfrak{h}$. Note that if $\mathfrak{h} \cong \mathfrak{g}$, then $C_*(\mathfrak{h}) \cong C_*(\mathfrak{g})$.

Next, consider the local dgla on an open $U \subset \mathbb{R}$, given by

$$\mathcal{L}(U) = \Omega_c(U) \otimes \mathfrak{g} \tag{11}$$

where the bracket is the the lie bracket on $\mathfrak{g}$, and the differential is the de Rham differential on forms. The *factorization envelope* is the prefactorization algebra defined by

$$\mathbb{U}\mathfrak{g} : U \mapsto H_*(C_*\mathcal{L}(U)). \tag{12}$$

Note that any factorization algebra of this type is locally constant, since the Poincare lemma says that $\Omega_c(U) \to \Omega_c(V)$ is a quasi-isomorphism for an inclusion of contractible open sets $U \subset V$, that is

$$\Omega_c(U) \simeq \mathbb{C}[-1] \tag{13}$$

Thus, as vector spaces

$$C_*(\Omega_c(U) \otimes \mathfrak{g}) \cong C_*(\mathfrak{g}[-1]) = \mathrm{Sym}(\mathfrak{g})[0] \tag{14}$$

We claim that $\mathbb{U}\mathfrak{g}(\mathbb{R}) = U\mathfrak{g}$. To show this, we construct map of lie algebras, $\Phi : \mathfrak{g} \to C_*(\Omega^*_c(\mathbb{R}) \otimes \mathfrak{g})$, sending $X \mapsto \epsilon_a \otimes X$, where $\epsilon_a \in \Omega^1_c(\mathbb{R})$ satisfies $\int \epsilon = 1$ and is a smooth bump form centered at a, and has support in some small interval $(a - \delta, a + \delta)$. Then, we can check that $\epsilon_1 - \epsilon_{-1} = dh$ where $h \in \Omega^0_c(\mathbb{R})$, and $h = -1$ near 0. then

$$\begin{aligned}(d_{DR} + d_{CE})(\epsilon_0 \otimes X)(h \otimes Y) =& (\epsilon_0 \otimes X)(d_{DR}h \otimes Y) + \epsilon_0 h \otimes [X, Y] \\ =& (\epsilon_0 \otimes X)((\epsilon_1 - \epsilon_{-1}) \otimes Y) - \epsilon_0 \otimes [X, Y] \\ =& \Phi(X)\Phi(Y) - \Phi(Y)\Phi(X) - \Phi([X, Y])\end{aligned} \tag{15}$$

So $\mathbb{U}\mathfrak{g}(\mathbb{R}) = U\mathfrak{g}$. In this way, we find that the enveloping algebra $U\mathfrak{g}$ is naturally constructed from the local dgla of $\mathfrak{g}$-valued forms on intervals. In general, for any dgla $\mathfrak{g}$, the map

$$\mathbb{U}\mathfrak{g} : U \mapsto C_*(\Omega^*_c(U) \otimes \mathfrak{g}) \tag{16}$$

defines a locally constant factorization algebra on any manifold M, called the *factorization envelope* of $\mathfrak{g}$.

2. Factorization Algebras

The data of a prefactorization algebra defines a precosheaf $U \mapsto \mathcal{F}(U)$. For this to define a factorization algebra, we need this precosheaf to be a cosheaf with respect to a special topology.

Recall, $\mathcal{F}$ is an ordinary cosheaf on X if for any open set $U \subset X$ and cover $\{U_i\}_{i \in I}$ of U that the following sequence

$$\bigoplus_{i,j} \mathcal{F}(U_i \cap U_j) \to \bigoplus_k \mathcal{F}(U_k) \to \mathcal{F}(U) \to 0 \tag{17}$$

is exact.

A prefactorization algebra $\mathcal{F}$ is a factorization algebra if $\mathcal{F}$ satisfies the cosheaf property above for a more restrictive class of covers on X.

Definition 2.1. An open cover $\{U_i\}_{i\in I}$ of U is a *Weiss* cover, if for every finite collection of points $\{x_j\} \subset U$, there exists a element U_k of the cover such that $\{x_j\} \subset U_k$.

Remark 2.2.
(1) Suppose $\{U_i\}_{i\in I}$ is a Weiss cover of U. Then, the collection $\{U_i^{\times n} \subset U^{\times n}\}_{i\in I}$ is an open cover (in the ordinary sense) of the product space $U^{\times n}$. This implies that a Weiss cover on X induces a topology on the Ran space, the collection of all finite subsets of X, $\mathrm{Ran}(X)$. This leads to another characterization of a factorization algebra: a factorization algebra on X is equivalent to a *cosheaf* on the Ran space.
(2) Weiss covers, in general, are huge. For instance, $\{\mathbb{R} \setminus q \mid q \in \mathbb{Q}\}$ form a Weiss cover of $\mathbb{R}$.

Definition 2.3. A *factorization algebra* on X is a prefactorization algebra $\mathcal{F}$ on X that satisfies:

- for any open set $U \subset X$ and Weiss cover $\{U_i\}$ of U, the sequence (17) is exact;
- for any $U \sqcup V \subset X$ the natural map
$$\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \sqcup V)$$
is an isomorphism.

All our examples so far will be either of the form $U \to \mathrm{Sym}(\mathcal{E}_c(U))$, where $\mathcal{E}_c$ is the space of compactly supported section of a graded vector bundle, or deformations thereof. These examples are automatically multiplictative factorization algebras, due to the compactly supported nature, and the multiplicative behavior of the ring of functions, e.g.

(18)
$$\mathrm{Sym}(\Omega_c(U \amalg V)) = \mathrm{Sym}(\Omega_c(U) \oplus \Omega_c(V)) = \mathrm{Sym}(\Omega_c(U)) \otimes \mathrm{Sym}(\Omega_c(V))$$

We note that to build locally constant factorization algebras on S^1, we assign the factorization algebra A^{fact} to $S^1 - \{0\} \cong \mathbb{R}$, and then are attach an $A - A$ bimodule M to an interval containing 0. The defect at the point captures the monodromy of the factorization algebra around the circle.

2.0.1. *Gluing up to homotopy.* For factorization algebras valued in chain complexes Ch, it is more natural to use the homotopy cosheaf condition. Let $\mathcal{F}$ be a precosheaf valued in Ch, and $\mathcal{U} = \{U_i\}$ an open cover for U. Define the Čech complex $\check{C}(\mathcal{U}, \mathcal{F})$ as follows. As a graded vector space it is

(19)
$$\bigoplus_{k=1}^{\infty} \left(\bigoplus_{j_1,\ldots,j_k \in I j_i \text{ distinct}} \mathcal{F}(U_{j_1} \cap \cdots U_{j_k})[k-1] \right).$$

The differential is given by the totalization of the internal differential $\mathrm{d}_{\mathcal{F}}$ on $\mathcal{F}$ and the Čech differential d_{Cech}.

Definition 2.4. A factorization algebra on X valued in chain complexes is a prefactorization algebra $\mathcal{F}$ on X valued in chain complexes satisfying:

(1) for every open $U \subset X$ and Weiss cover $\mathcal{U}$ of U the natural map

$$\check{C}(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(U) = \int_U \mathcal{F}$$

is a quasi-isomorphism;

(2) for any $U \sqcup V \subset X$ the natural map

$$\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \sqcup V)$$

is a quasi-isomorphism.

Remark 2.5. The Čech complex $\check{C}(\mathcal{U}, \mathcal{F})$ is nothing but a model for the homotopy colimit

$$\operatorname*{hocolim}_{U \in \mathcal{U}} \mathcal{F}(U).$$

2.1. **Locally constant factorization algebras.** We have formulated the notion of a factorization algebra as an algebra over a colored operad of disjoint disks Disj_M satisfying a certain gluing condition.

A simplification of the above operad appears when we consider locally constant factorization algebras.

Let E_n be the operad (pseudo-tensor category with one object) of little n-disks. The k-ary morphisms are given by the the space of framed embeddings of disjoint disks in $\mathbb{R}^n$

$$\sqcup_{i \leq k} D^n \hookrightarrow D^n.$$

Composition is defined in the obvious way. E_n objects in vector spaces or chain complexes are referred to as E_n-algebras. The generalization of the earlier result equating locally constant factorization algebras on $\mathbb{R}$ and associative algebras, is due to Lurie:

Theorem 1 (Lurie [Lur]). There is an equivalence of $(\infty, 1)$-categroies

$$Fact_{l.c.}(\mathbb{R}^n) \simeq E_n{-}Alg \tag{20}$$

Note that commutative algebras give factorization algebras on every $\mathbb{R}^n$, since they are naturally E_∞-algebras.

3. Factorization Homology

We can push-forward a factorization algebra $\mathcal{F}$ over a continuous map $p : X \to Y$,

$$(p_* \mathcal{F})(U) = \mathcal{F}(p^{-1}(U)), \qquad U \subset Y. \tag{21}$$

We define the *factorization homology*, $\int_X \mathcal{F}$, for any factorization algebra $\mathcal{F}$, as push-forward to a point $f : X \to \mathrm{pt}$, or equivalently taking global sections

$$\int_X \mathcal{F} = \mathcal{F}(X). \tag{22}$$

We explain just some simple examples of how to calculate $\int_X \mathcal{F}$.

3.1. Hochschild homology.

Consider the factorization algebra $\mathcal{F}_{(M,A,N)}$ on $[0,1]$, which assigns the associative algebra A to opens of the form (a,b), assigns the right A-module M to opens of the form $[0,a)$, and assigns the left A module N to opens $(a,1]$. We then have the following technical lemma, due to Gwillam ([Gwi12])

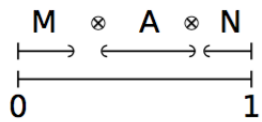
$$\mathcal{F}_{(M,A,N)}([0,1]) = M \otimes^{\mathbb{L}}_A N \tag{23}$$

This is proven using the cosheaf Cech description of factorization homology. However, we can gain some insight into this result by considering the Weiss cover given by the open sets $U_x = [0,1]/\{x\}$. Its easy to see that

$$\mathcal{F}_{(M,A,N)}(U_{x_0}) = M \otimes N \tag{24}$$

and

$$\mathcal{F}_{(M,A,N)}(U_{x_0} \cap U_{x_1}) = M \otimes A \otimes N \tag{25}$$



All the possible maps give the contributions to the differentials of the bar complex. E.g. the first structure map in the sequence

$$\mathcal{F}_{(M,A,N)}(U_{x_0} \cap U_{x_1}) \to \mathcal{F}_{(M,A,N)}(U_{x_0}) \oplus \mathcal{F}_{(M,A,N)}(U_{x_1}) \to \mathcal{F}_{(M,A,N)}([0,1]) \tag{26}$$

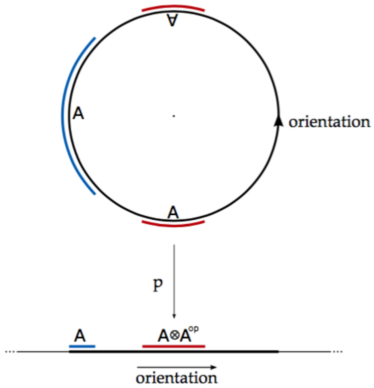
is

$$m \otimes a \otimes n \mapsto (ma \otimes n, -m \otimes an) \tag{27}$$

All the maps in the construction of the 'bar' complex construction of $M \otimes^{\mathbb{L}}_A N$ appear in this way

Now, consider the locally constant factorization algebra A^{fact} on S^1, built from an associative algebra A. We push-forward this factorization algebra with the map $p = \sin : S^1 \to [-1,1]$. Then $p_* A^{fact}$ is characterized by $p_* A^{fact}((a,b)) = A \otimes A^{op}$, and we can see that

$$p_* A^{fact} = \mathcal{F}_{(A, A\otimes A^{op}, A)} \tag{28}$$



Hence, we have

$$A^{fact}(S^1) = p_* A^{fact}([-1,1]) \\ = \mathcal{F}_{(A, A\otimes A^{op}, A)}([-1,1]) \\ = A \otimes^{\mathbb{L}}_{A\otimes A^{op}} A \\ \cong HH_*(A) \tag{29}$$

Insead, we could consider the factorization algebra on S^1 where we have glued the ends of $\mathbb{R} = S^1 - \{0\}$ together using the twisted bimodule A_σ at $\{0\}$. In this case we would find

$$A_\sigma^{fact}(S^1) = A \otimes^{\mathbb{L}}_{A\otimes A^{op}} A_\sigma \cong HH_*(A, A_\sigma) \tag{30}$$

Lastly, we use the de Rham description of factorization homology to compute the Hochshild holomogy of the enveloping algebra. We know that

$$\mathbb{U}\mathfrak{g}(S^1) = HH_*(U\mathfrak{g}). \tag{31}$$

This follows from the Cech computations above. However, we can also perform this calculation using the de Rham model. We begin by noting that S^1 is formal, i.e. $\Omega^*(S^1) \cong H^*(S^1) = \mathbb{C}[0] \oplus \mathbb{C}[-1]$ as dg assoc algebras, since there are no higher operations in cohomology (Massey products). Thus, $C_*(\Omega^*(S^1) \otimes \mathfrak{g}) \cong C_*(\mathfrak{g} \oplus \mathfrak{g}[-1]) = C_*(\mathfrak{g}) \otimes \mathrm{Sym}(\mathfrak{g})$ as chain complexes. As before, we can easily check that the $\mathfrak{g}$-module action on $\mathrm{Sym}(\mathfrak{g})$ agrees with that of $U\mathfrak{g}$, and we find that the Hochshild homology of $U\mathfrak{g}$ computes the Lie algebra homology of the $U\mathfrak{g}$ as a $\mathfrak{g}$-module,

$$HH_*(U\mathfrak{g}) = H_*(C_*(\mathfrak{g}) \otimes U\mathfrak{g}) = H_*(\mathfrak{g}, U\mathfrak{g}). \tag{32}$$

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